Irregularity in strong approximation and the multilevel Monte Carlo method

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Outline

Take random variables $X, \hat{X} : \Omega \to \mathbb{R}$.

- Try to determine the error $\left\| g(X) - g(\hat{X}) \right\|_p$ in terms of $\left\| X - \hat{X} \right\|_p$

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Indicator functions

Theorem 1

Let $0 < p < \infty$ and suppose that X is a random variable with a bounded density f_X. Then for all $K \in \mathbb{R}$ and for all random variables $\hat{X}$ we have

$$\left\| \chi_{[K,\infty)}(X) - \chi_{[K,\infty)}(\hat{X}) \right\|_1 \leq 3(\sup f_X)^{\frac{p}{p+1}} \left\| X - \hat{X} \right\|_p^{\frac{p}{p+1}},$$

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where $\frac{p}{p+1}$ is the optimal exponent.

Functions of bounded variation on $\mathbb{R}$

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- The total variation measure $|\mu| = \mu_1 + \mu_2$.

Theorem 2

Let $1 \leq q < \infty$ *and* $g \in BV$, *and suppose that* X *has a bounded density. Then*

$$\left\| g(X) - g(\hat{X}) \right\|_q^q \leq 3^{q+1} V(g)^q (\sup f_X \vee \sqrt{\sup f_X}) \left\| X - \hat{X} \right\|_p^{\frac{p}{p+1}}$$

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Extension

Idea:

$$\left\|\chi_{[K,\infty)}(X) - \chi_{[K,\infty)}(\hat{X})\right\|_1 = \mathbb{P}(X \geq K, \hat{X} < K) + \mathbb{P}(X < K, \hat{X} \geq K)$$

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Space BV_φ

Consider the collection of set functions $\mu : \{$ bounded Borel sets $\} \to \mathbb{R}$,

$$\mu(F) := \mu^1(F) - \mu^2(F),$$

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Then $g \in BV_\varphi$ if

$$g(x) = \begin{cases} \mu([0,x)), & x > 0, \\ \mu([x,0)), & x \leq 0 \end{cases}$$

for some μ such that

$$\|g\|_\varphi := \int_{\mathbb{R}} \varphi(z)\, d|\mu|(z) < \infty.$$

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Theorem 3

Let $1 \le q < \infty$, $0 < \theta < 1$, and $g \in BV_\varphi$, and suppose that X has a bounded density. Then

$$\left\| g(X) - g(\hat{X}) \right\|_q^q \le C(p, X, \theta) \, \|g\|_{\varphi^{\theta/q}}^q \left\| X - \hat{X} \right\|_p^{\frac{p}{p+1}(1-\theta)}$$

for any $1 \le p < \infty$.

Application to SDEs

Take $X = X_T$, the solution of the 1-dimensional SDE

$$\begin{cases} X_0 = x_0, \\ dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \ t \in [0, T], \end{cases}$$

where $x_0 \in \mathbb{R}$, $\sigma, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and W is a standard BM.

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Take $\hat{X} = X_T^{E,n}$, the Euler approximation of X_T relative to π_n , i.e.

$$X_0^{E,n} = x_0, \text{ and}$$
$$X_{t_{i+1}}^{E,n} = X_{t_i}^{E,n} + b(t_i, X_{t_i}^{E,n})(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^{E,n})(W_{t_{i+1}} - W_{t_i}).$$

Assumptions

Assume that $\sigma, b \in C([0,T] \times \mathbb{R})$ and for $f \in \{\sigma, b\}$ there exists a constant C_T such that

1) $|f(t,x) - f(t,y)| \leq C_T |x - y|$
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Sufficient conditions for the assumption 3):

- uniform ellipticity
- Result of Caballero, Fernández, Nualart (1998)

Result for BV

Theorem 4

Let $1 \leq p < \infty$ and $g \in BV$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

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where M depends on x_0, T and the Lipschitz coefficients.

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- In the case of bounded coefficients, we can show the rate $\frac{1}{2} - \frac{M}{(-\log|\pi_n|)^{1/2}}$ for functions g of polynomial variation

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- Consider the combined estimator $\hat{Y} = \sum_{l=0}^{L} \hat{Y}_l$

Multilevel Monte Carlo Method

Theorem 5 (Giles 2008)

Suppose there are positive constants $\alpha \geq \frac{1}{2}$, β, c_1, c_2 *such that*

i) $|\mathbb{E}g(X_T^{E,h_l}) - \mathbb{E}g(X_T)| \leq c_1 h_l^{\alpha}$,

ii) $Var(g(X_T^{E,h_l}) - g(X_T^{E,h_{l-1}})) \leq c_2 h_l^{\beta}$.

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Then for any $\varepsilon < 1/e$ *the multilevel estimator* $\hat{Y}$ *has*

$$MSE(\hat{Y}) < \varepsilon^2$$

and

$$C(\hat{Y}) \leq \begin{cases} c_3 \varepsilon^{-2}, & \beta > 1, \\ c_3 \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_3 \varepsilon^{-2-\frac{(1-\beta)}{\alpha}}, & 0 < \beta < 1. \end{cases}$$

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Parameter α

We take $\alpha = 1$ in

i) $|\mathbb{E}g(X_T^{E,h_l}) - \mathbb{E}g(X_T)| \leq c_1 h_l^\alpha$

by weak convergence results:

- Talay, Tubaro (1990): $g \in C_{pol}^\infty$, under the assumption $\sigma, b \in C_b^\infty$
- Bally, Talay (1996): g measurable and bounded, under uniform hypoellipticity
- Guyon (2006): extension to measurable functions with exponential growth, under uniform ellipticity

Parameter β

We have to determine β in $Var(g(X_T^{E,h_l}) - g(X_T^{E,h_{l-1}})) \leq c_2 h_l^{\beta}$.

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If g is Lipschitz, then $\beta = 1$.

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Our results imply that $\beta = \frac{1}{2} - \frac{A}{((l \log 2) \vee B)^{1/2}}$.

Results applied to the Giles' method

Corollary 6

Assume that $\alpha = 1$ *and* $\beta = \frac{1}{2} - \frac{A}{((l \log 2) \vee B)^{1/2}}$, *and*

- $g \in BV$ *and standard assumptions for the SDE, or*
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$$MSE(\hat{Y}) < \varepsilon^{2-\delta(\varepsilon)},$$

where $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *and*

$$C(\hat{Y}) \leq c_4 \varepsilon^{-2-\frac{(1-1/2)}{1}} = c_4 \varepsilon^{-2.5}.$$

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