Weighted Monte Carlo method applied to acceleration oriented traffic flow model

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Introduction

Main macro characteristics of the Vehicular Traffic Flow (VTF) are:

1. flux, flow Q [auto / time]
2. density C [auto / length]
3. mean velocity V [km/hour]

Approaches to modelling: *deterministic and probabilistic (stochastic)***.**

– In the bases of the deterministic approach lies a functional relation between some of the main VTF characteristics, such as, e.g. velocity and the distance between the cars in the VTF.

– In stochastic models, oh the other hand, the VTF is considered as a probabilistic process.

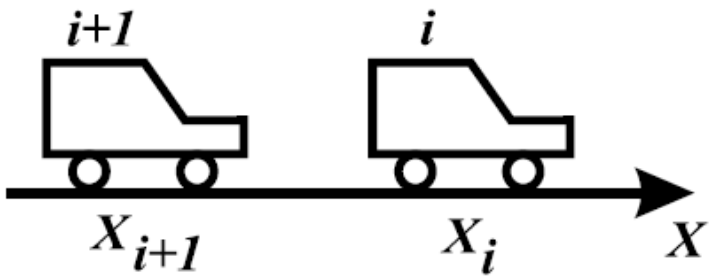
All VTF models can be subdivided into three groups:

- Analog models. VTF is likened to a physical flow: fluid dynamic flow (**macroscopic** models) and gas dynamic flow (**mesoscopic** or **kinetic** models).
- **Microscopic** (cellular automata, car following model) models. These models are based on the assumption that there exists a relation between the movement of the leading and the following cars.

VTF MODELS

1) Microscopic models (cellular automata, car following model)

This type of models regards the VTF as a system of interacting particles. Moreover, these models use Newton equations of motion, written individually for each particle in the system.



2) Macroscopic models (fluid dynamic)

Within this type of models VTF is regarded as a one-dimensional flow of compressible fluid.
We assume that.

1. The flow conservation low is valid for VTF (this condition is stated through the continuity equation; the on- and off-ramps are taken into account)

$$\frac{\partial c(x;t)}{\partial t} + \frac{\partial J(x;t)}{\partial x} = \sum_{i=1}^{J_{in}} \alpha_i(x - x_i;t) - \sum_{j=1}^{J_{out}} \beta_j(x - x_j;t)$$

2. There exists a one-to-one relation
between the flux and the density of the VTF
or
between the mean velocity and the density of the VTF

3) Kinetic

Within the kinetic theory VTF is considered as a gas, i.e. a system of interacting particles; every particle in this system corresponds to a certain vehicle. These models are called mesoscopic due to the fact that they use information of the single vehicle behaviour as input and produce results of the whole traffic flow.

The most obvious differences of VTF from gas flow are

- VTF is organized and mostly one dimensional;
- based on a deterministic set of rules;
- depends (more or less) on individual drivers.

Boltzmann kinetic equation:

$$\left[\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\nabla_r + \boldsymbol{F}\cdot\nabla_p\right] f(\boldsymbol{r},\boldsymbol{p};t) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

equation describes the probability distribution.

$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$ this item is responsible for particle interactions in the system. We assume, that the state of a vehicle is determined by a number of characteristics. So, **interaction** of 2 vehicles means the **event**, which leads to any change in the state of these vehicles.

Acceleration oriented VTF model *(K.T. Waldeer)*

By analogy with gas kinetic transport theory, the state of a car is determined by a space coordinate and velocity in most mesoscopic VTF models. In this type of models the acceleration is a jump process in the velocity variable.

In the model, suggested by K.T. Waldeer, the acceleration variable is regarded as one of the phase coordinates, describing the state of a car. This modification leads to the discontinuous change of the acceleration variable as a result of the interaction in the system (not velocity variable, as it happens in most of kinetic models).

For a **spatial homogeneous** traffic flow the Boltzmann-like equation (BE) describing VTF has the following form

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial v} = \int\limits_{\bar{v},\bar{a},a'} (\Sigma(a \mid v, a', \bar{v}, \bar{a}) f(a', v, t) - \\ -\Sigma(a'|v, a, \bar{v}, \bar{a}) f(a, v, t))\, f(\bar{a}, \bar{v}, t) d\bar{v}\, d\bar{a}\, da' \ .$$

Σ is weighted interaction rate function *(which determines the type of interaction in the system),*

$f(v, a, t)$ is *a one-particle probability density, describing the state of a car.*

Boundary condition:

1) There are no cars with negative velocities (i.e. there are no cars with negative acceleration among the cars with V=0);
2) There is a maximum velocity of a VTF, which can not be exceeded (i.e. there are no cars with positive acceleration among the cars with $V=V_{max}$).

Initial velocity density without any acceleration:

$$f(v,a,t=0)=\frac{1}{\sqrt{2\pi}\sigma_0}e^{\frac{(v-V_0)^2}{2\sigma_0^2}}\delta(a)$$

Numerical Solving Procedure (K.T.Waldeer)

We solve the equation with the help of the following splitting the movement and the interaction process

$$f(v,a,t+\Delta t)=(1-\mathcal{D}\Delta t+\mathcal{J}\Delta t)f(v,a,t)+\mathcal{O}(\Delta t)$$
$$=(1-\mathcal{D}\Delta t)(1+\mathcal{J}\Delta t)f(v,a,t)+\mathcal{O}(\Delta t)$$

in 2 parts executed consecutively in a time interval Δt:
Here $\mathcal{J}$ is a nonlinear operator on the right side of BE, describing the discontinuous stochastic acceleration change, $\mathcal{D}=a\partial/\partial v$ is a velocity drift operator on the left side of BE.

Splitting process

The transition of a state probability density *f* to a new state at $t+\Delta t$ is made in 2 steps:

1) First, in a given time interval Δt density *f* is changed in the acceleration variable resulting in a new, intermediate probability density $f_I(v,a,t)=(1+\mathcal{J}\Delta t)f(v,a,t)$

2) Then this intermediate probability density f_I is changed in velocity due to the drift operator $\mathcal{D}=a\partial/\partial v$ into a new state probability density at $t+\Delta t$

Simulation process:

1) Choose the integration time step Δt
2) For each of *i=1…N cars do*
   a) *Choose a leading car j*
   b) *Calculate a new acceleration for car number i* a_i^*
3) For each of *i=1…N* cars calculate new velocities $(v_i+a_i^*\Delta t)\rightarrow v_i$
4) The next time step.

Integral equation of the second kind

Density $f(a,v,t)$ satisfies the Boltzmann type equation:

$$\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial v}=\int\limits_{\bar v,\bar a,a'}\left[\Sigma(a|v,a',\bar v,\bar a)f(a',v,t)-\Sigma(a'|v,a,\bar v,\bar a)f(a,v,t)\right]f(\bar a,\bar v,t)\,\mathtt{d}\bar a\,\mathtt{d}\bar v\,\mathtt{d}a' \tag{1}$$

We obtain an integral equation of the second kind equivalent to (1) to eliminate the parameter Δt from the simulation procedure.

Let us introduce the p.d.f. of N-particles system $P(A,V,t)=P(a_1,v_1,\ldots,a_N,v_N,t)$:

$$\frac{\partial P}{\partial t}+A\frac{\partial P}{\partial V}+\nu(A,V)P(A,V,t)=J_N(A,V,t),\ \text{here}$$

$$\nu(A,V)=\frac{1}{N-1}\sum_{i\neq j}\int\Sigma(a_i\to a_i''|v_i,v_j,a_j)\,\mathtt{d}a_i''=\sum_{i\neq j}\frac{\nu(a_i,v_i,v_j,a_j)}{N-1}=\sum_{\pi}\frac{\nu(\pi)}{N-1}.$$

$$J_N(A,V,t)=\int F(A'\to A|V)P(A',V,t)\,\mathtt{d}A'=$$

$$=\frac{1}{N-1}\sum_{i\neq j}\int\left[\Sigma(a_i'\to a_i|v_i,a_j,v_j)\prod_{m\neq i,m=1}^{N}\delta(a_m'-a_m)\right]P(A',V,t)\,\mathtt{d}A'.$$

It is reasonable to use collision density $\Phi(A,V,t)=\nu(A,V)P(A,V,t)$ rather than function P for simulating the Markov chain. Moreover, it is even more convenient to use function $\Psi(A,V,t)$:

$$\Phi(A,V,t)=\int\limits_0^t\int\Psi(A,V',t')K_t(t'\to t|A,V')\,\mathtt{d}t'K_V(V'\to V|A,t-t')\,\mathtt{d}V',$$

In this case the free term is nonzero only for $t=0$:

$$\Psi(A,V,t)=\delta(t)P_0(A,V)+$$

$$\iiint\limits_0^t\Psi(A',V',t')K_t(t'\to t|A',V')\,\mathtt{d}t'K_V(V'\to V|A',t-t')\,\mathtt{d}V'K_A(A'\to A|V)\,\mathtt{d}A',$$

the transition densities are

$$K_t(t'\to t|A',V')=\chi(t'<t)\nu(A',V'+A'(t-t'))\exp\left\{-\int\limits_{t'}^t\nu(A',V'+A'(\tau-t'))\,\mathtt{d}\tau\right\},$$

$$K_V(V'\to V|A',t-t')=\delta(V-V'-A'(t-t')),\;\; K_A(A'\to A|V)=\frac{F(A'\to A|V)}{\nu(A',V)}.$$

The transitions from the state (A',V',t') to the state (A,V,t) is performed as follows

$$(A',V',t')\to(A',V',t)\to(A',V,t)\xrightarrow{\pi}(A,V,t).$$

Stratification according to the pair number

We add the index π of the pair of cars participating in the current interaction in the system to the phase space. Function Ψ can be represented in the form:

$$\Psi(A, V, t) = \sum_{\pi} F(\pi, A, V, t).$$

We obtain the integral equation for the function F in the modified phase space $(Z, t) = (\pi, A, V, t)$:

$$F(Z, t) = \delta(t) P_0(A, V) \delta(\pi_0) + \int_0^t \int F(Z', t') K(Z', t' \to Z, t) \, \mathtt{d}Z' \, \mathtt{d}t'.$$

Here π_0 is an arbitrary pair number, and the kernel K is the factorization of the transition densities:

$$K(Z', t' \to Z, t) = K_t(t' \to t | A', V') K_V(V' \to V | A', t - t') K_\pi(\pi) K_a(a'_i \to a_i | \pi, V).$$

Estimation of the functionals

Usually the following linear functionals are of interest

$$I_h(T) = \iint h(v,a) f(T,v,a)\, \mathtt{d}v\, \mathtt{d}a.$$

One can show that [Mikhailov, Rogasinsky]

$$I_h(T) = \iint\limits_0^T H(A, V+A(T-t')) \exp\left\{ -\int\limits_{t'}^T \nu(A, V + A(\tau - t')\, \mathtt{d}\tau \right\} F(\pi, A, V, t') \mathtt{d}Z \mathtt{d}t',$$

here $H(V,A) = \frac{1}{N} \sum\limits_{i=1}^{N} h(v_i, a_i)$.

For estimation of the latter integral we can use standard collision estimator ξ or absorbtion one η:

$$\xi = \sum_{k=0}^{S} Q_k \tilde{H}(A_k, V_k, t_k, T),$$

$$\eta = Q_S \frac{\tilde{H}(A_k, V_k, t_k, T)}{p(A_k, V_k, t_k, T)}.$$

Majorant frequency method

Let $\nu_{max} \geq \nu(\varpi)$ for any $\varpi = (i,j)$ and $\nu^* = \sum\limits_{i \neq j} \nu_{max}/(N-1) = N \cdot \nu_{max}$. Then

$$\frac{\partial P}{\partial t} + A\frac{\partial P}{\partial V} + \nu^* P(A,V,t) = J_N^*(A,V,t) = \nu^* \int K^*(A' \to A|V) P(A',V,t)\, \mathtt{d}A',$$

$$K^* = \sum_{\pi=(i,j)} K_\pi^* \left[\left(1 - \frac{\nu'(\pi)}{\nu_{max}}\right) \delta(a_i' - a_i) + \frac{\nu'(\pi)}{\nu_{max}} \frac{\Sigma(a_i' \to a_i | v_i, v_j, a_j')}{\nu'(\pi)} \right] \prod_{m \neq i} \delta(\ldots).$$

$K_\pi^* = [N(N-1)]^{-1}$ is equiprobability distribution of the pair number π of cars participating in interaction. Time distribution transforms to the exponential one:

$$K_t^*(t' \to t) = \chi(t' < t) \nu^* \exp\left\{-\nu^*(t-t')\right\},$$

For the function $F^*(Z,t)$ we obtain an integral equation with the kernel

$$K(Z', t' \to Z, t) = K_t^*(t' \to t) K_V(V' \to V | A', t-t') K_\pi^* K_a(a_i' \to a_i | \pi = (i,j), V).$$

$$I_h(T) = \iint_0^T H(A, V + A(T-t')) \exp\left\{-\nu^*(T-t')\right\} F^*(\pi, A, V, t')\, \mathtt{d}Z\, \mathtt{d}t'.$$

Model Cases Compared with Analytical Solutions

We consider all the cars to have bounded velocities $0 \leq v_i \leq V_{max}$ and a single value acceleration $\pm a_0$.

We tested two interaction types to compare with the analytical solutions.

1) Maxwell interaction

$$\Sigma(a_i' \to a_i | v_i, a_j, v_j) = \frac{1}{\mathcal{T}} \begin{cases} \delta(a - a_0), & v_i \leq v_j, \\ \delta(a + a_0), & v_j < v_i. \end{cases}$$

In this case $\nu(\pi) = const = 1/\mathcal{T}$, $\nu^* = N/\mathcal{T}$. Under these conditions, kinetic equation can be solved analytically in stochastic equilibrium, resulting in

$$f(v, a) = \frac{\pi}{4\sqrt{3}\sigma_v} \cosh^{-2}\left\{\frac{\pi}{2\sqrt{3}} \frac{(v - V)}{\sigma_v}\right\} \frac{\delta(a - a_0) + \delta(a + a_0)}{2},$$

with the mean V and the variance $\sigma_v^2 = (\pi \mathcal{T} a_0)^2/3$. We used the following parameters in the computer simulation with $N = 1000$ stochastic cars and $M = 1000$ runs:
$\sigma_0^2 = 0.1$ m^2/s^2, $V = 20$ m/s,$a_0 = 0.3$ m/s^2, $T = 18$ s, $\mathcal{T} = 2$ s, $\sigma_v = 1.088$ m/s.

2) Hard sphere interaction

$$\Sigma(a_i' \to a_i | v_i, a_j, v_j) = r_0 |v_j - v_i| \begin{cases} \delta(a - a_0), & v_i \le v_j, \\ \delta(a + a_0), & v_j < v_i. \end{cases}$$

In this case $\nu(\pi) = r_0|v_i - v_j|$, $\nu_{max} = r_0 V_{max}$, $\nu^* = N V_{max} r_0$. The solution in stochastic equilibrium is given by

$$f(v, a) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left\{ -\frac{(v - V)^2}{2\sigma_v^2} \right\} \frac{\delta(a - a_0) + \delta(a + a_0)}{2},$$

with the mean V and the variance $\sigma_v^2 = a_0/r_0$. We used the following parameters in the computer simulation with $N = 1000$ stochastic cars and $M = 1000$ runs:
$\sigma_0^2 = 0.1$ m^2/s^2, $V = 20$ m/s, $a_0 = 0.3$ m/s^2, $T = 62$ s, $r_0 = 0.25$ m^{-1}, $\sigma_v = 1.095$ m/s.

References

[1] K.T. Waldeer. The direct simulation Monte Carlo method applied to a Boltzmann-like vehicular traffic flow model // Comp. Phys. Commun. 2003. V. 156, N 1, pp. 1–12.

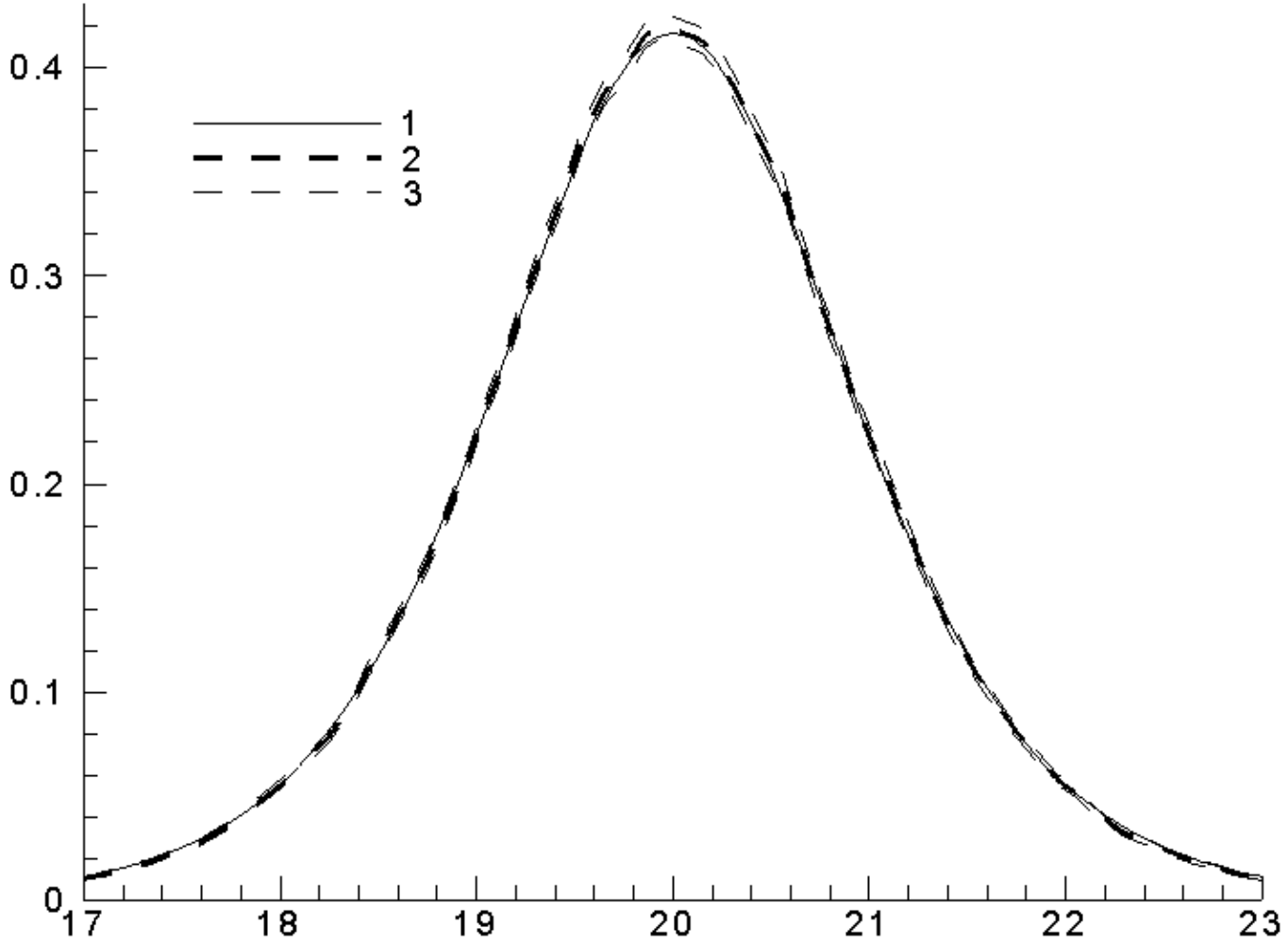


Figure 1: Estimation of $f(v)$ for the Maxwell interaction, $T = 18$ s:
1 – exact solution, 2 – estimation, 3 – confidence interval $\pm 3\sigma$.

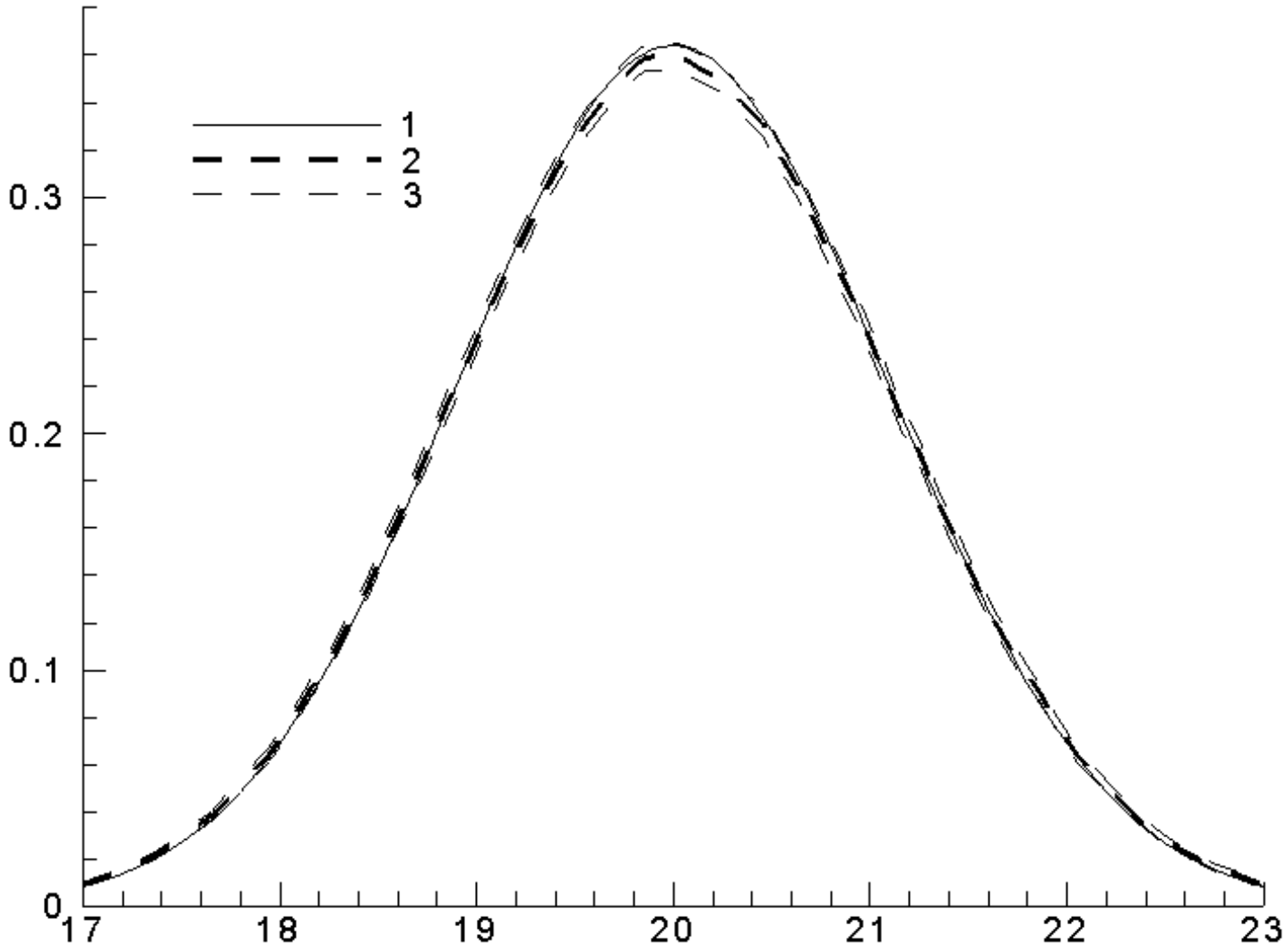


Figure 2: Estimation of $f(v)$ for the hard sphere interaction, $T = 62$ s:
1 – exact solution, 2 – estimation, 3 – confidence interval $\pm 3\sigma$.