A construction of digital nets in $\mathbb{R}^s$

Josef Dick

School of Mathematics and Statistics, UNSW, Sydney, Australia
Blog: http://quasirandomideas.wordpress.com

The task is to approximate an integral

$$I_s(f) = \int_\Omega f(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z}$$

for some integrand f and domain $\Omega \subseteq \mathbb{R}^s$ by some quadrature rule

$$Q_{N,s}(f) = \sum_{n=1}^{N} \lambda_n f(\boldsymbol{x}_n)$$

at some sample points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \Omega$.
We focus on $\Omega = \mathbb{R}^s$.

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Questions:

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2. How should we choose the quadrature points $\boldsymbol{x}_n$?
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In particular:

(a) Try to avoid using a transformation $[0,1]^s \to \mathbb{R}^s$.

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Aim:

- Answer questions (1-3) such that (a-c) is satisfied.

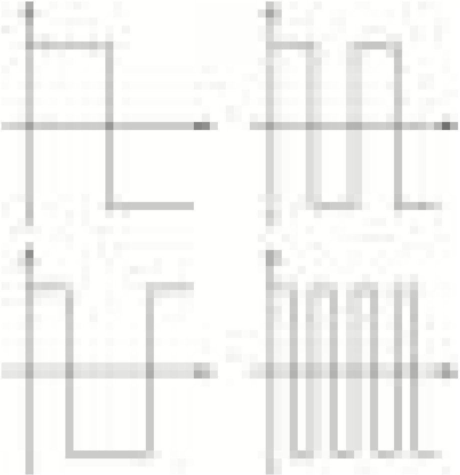
Walsh function

Walsh functions on $[0,1)$:
Use base 2 expansions:

$$\begin{aligned} k &= \kappa_0 + \kappa_1 2 + \cdots + \kappa_{a-1} 2^{a-1} \\ x &= x_1 2^{-1} + x_2 2^{-2} + \cdots \end{aligned}$$

Then

$$\mathrm{wal}_k(x) = (-1)^{\kappa_0 x_1 + \cdots + \kappa_{a-1} x_a}.$$



Walsh model (Coifman, Thiele,...)

Extend to $\mathbb{R}^s$: let $j, l \in \mathbb{Z}$ and set

$$w_{j,k,l}(x) = 2^{-j/2} \mathrm{wal}_k(2^{-j}x - l)$$

The function $w_{j,k,l}$ has support $[2^j l, 2^j(l+1))$.

The system

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- Haar functions on $[0, 1)$: choose $k = 1$:

$$w_{0,0,0} \text{ and } w_{j,1,l} : j < 0, 0 \leq l < 2^{-j}$$

Tiles

Tiles $T_{j,k,l}$**:** We associate an elementary rectangle of area one in $\mathbb{R} \times \mathbb{R}_+$ to each function $w_{j,k,l}$. (Here $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.)

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For $j, l \in \mathbb{Z}$, $k \in \mathbb{N}_0$:

$$w_{j,k,l} \leftrightarrow T_{j,k,l} = [2^j l, 2^j(l+1)) \times [2^{-j}k, 2^{-j}(k+1))$$

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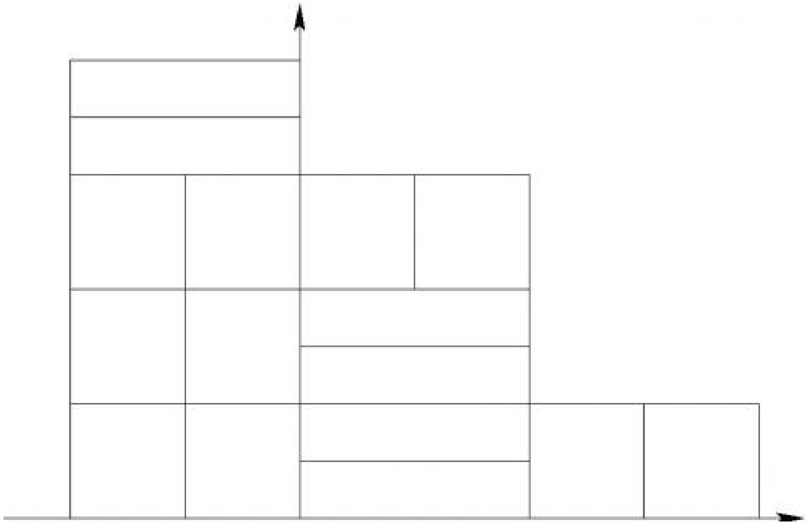
$$\cup_{j,l \in \mathbb{Z}} \cup_{k \in \mathbb{N}_0} T_{j,k,l} = \mathbb{R} \times \mathbb{R}_+$$

The Walsh phase plane



Complete orthonormal system

Orthogonal

$$\int_{\mathbb{R}} w_{j,k,l}(x) w_{j',k',l'}(x) \, \mathrm{d}x = 0 \Longleftrightarrow T_{j,k,l} \cap T_{j',k',l'} = \emptyset$$

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Complete

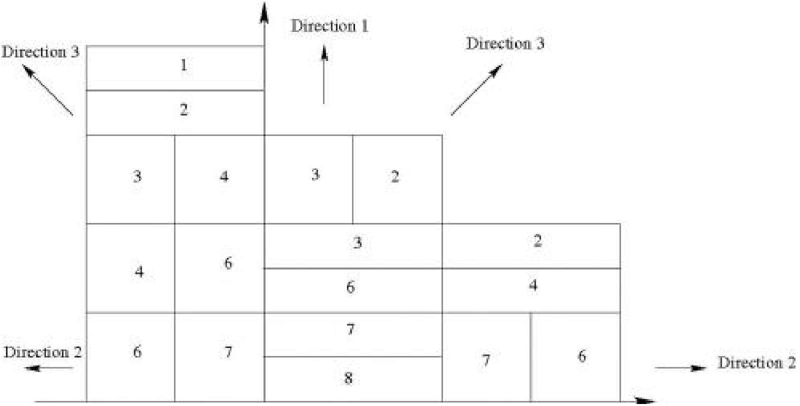
Let $B \subset \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{Z}$:

$$\{w_{j,k,l} : (j,k,l) \in B\} \iff \cup_{(j,k,l) \in B} T_{j,k,l} = \mathbb{R} \times \mathbb{R}_+$$

$f : \mathbb{R} \to \mathbb{R}$ Walsh coefficients:

$$\widehat{f}_{j,k,l} = \int_{\mathbb{R}} f(x) w_{j,k,l}(x) \,\mathrm{d}x$$

$$f(x) = \sum_{(j,k,l)\in B} \widehat{f}_{j,k,l} w_{j,k,l}(x)$$


Direction 1
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Smoothness of the integrand

3 types of 'smoothness' of the integrand $f : \mathbb{R} \rightarrow \mathbb{R}$:

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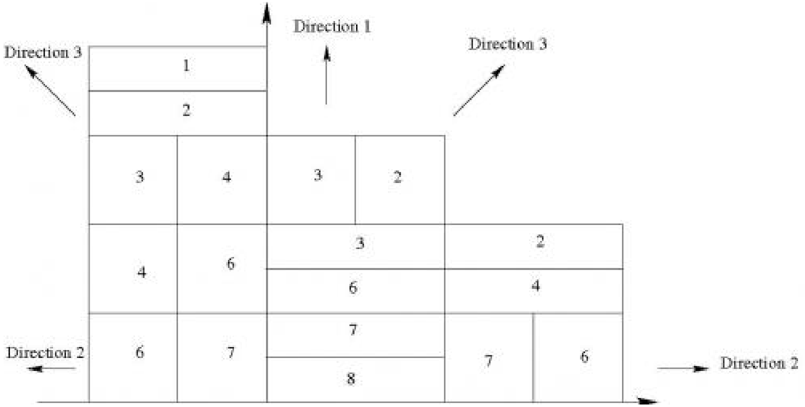
- Direction 2: how $f(x)$ decays as $|x| \to \infty$;
- Direction 3: how $f'(x)$ decays as $|x| \to \infty$;

Smoothness of the integrand



Numerical integration

Approximate $\int_{\mathbb{R}} f(x)\,\mathrm{d}x$ by

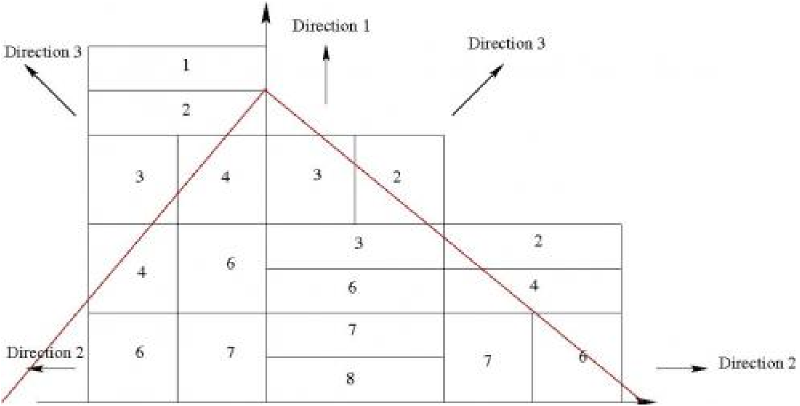
$$Q(f) = \sum_{n=0}^{N-1} \lambda_n f(x_n).$$

f has expansion

$$f(x) = \sum_{(j,k,l)\in B} \widehat{f}_{j,k,l} w_{j,k,l}(x).$$

Design $Q(f)$ such that all $w_{j,k,l}$ are integrated exactly for which $\widehat{f}_{j,k,l}$ are 'large'.

Numerical integration



Quadrature points in one dimension

- Use equally spaced points and equal weights in elementary intervals of the form $[2^j l, 2^j(l+1))$. $\Rightarrow$ **local structure**

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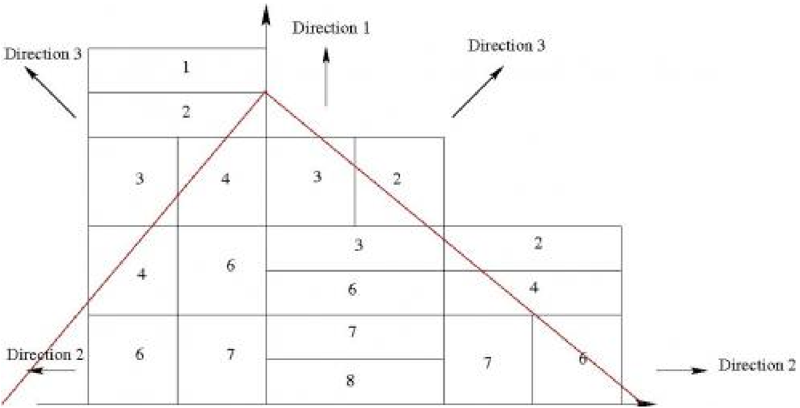
- Use more points where the coefficients decay slower as frequency k increases. $\Rightarrow$ **global structure**

Quadrature points in one dimension



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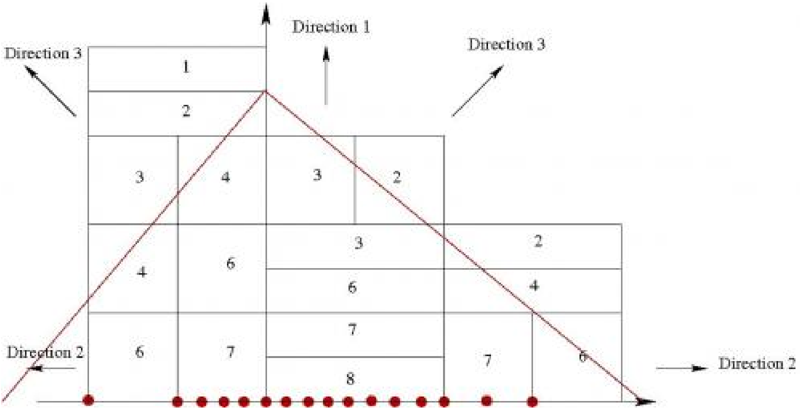
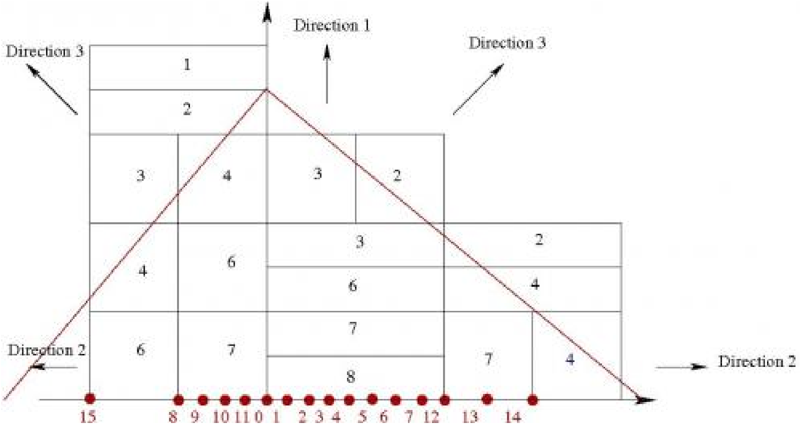


Quadrature points in high dimensions

Now extend this method to higher dimensions.

Step 1: number the points in each coordinate

Quadrature points in two dimensions



Construction of quadrature points

Step 2: Use digital net (Sobol, Faure, Niederreiter, Xing, Özbudak, Larcher, Schmid, Pillichshammer, ...)

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Then project digital net onto $\mathbb{R}^s$ using the numbering in each coordinate.
Let

$$z_{0,i}, z_{1,i}, \ldots, z_{2^m-1,i} \in \mathbb{R}$$

denote the points in the ith coordinate.

Let $C_i \in \mathbb{Z}_2$ denote the ith generating matrix. Let $n = n_0 + 2n_1 + \cdots + 2^{m-1} n_{m-1}$ and

$$C_i \begin{pmatrix} n_0 \\ \vdots \\ n_{m-1} \end{pmatrix} = \begin{pmatrix} \eta_{n,i,0} \\ \vdots \\ \eta_{n,i,m-1} \end{pmatrix}$$

Then set $\eta_{n,i} = \eta_{n,i,0} 2^{m-1} + \cdots + \eta_{n,i,m-1}$ and $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ where

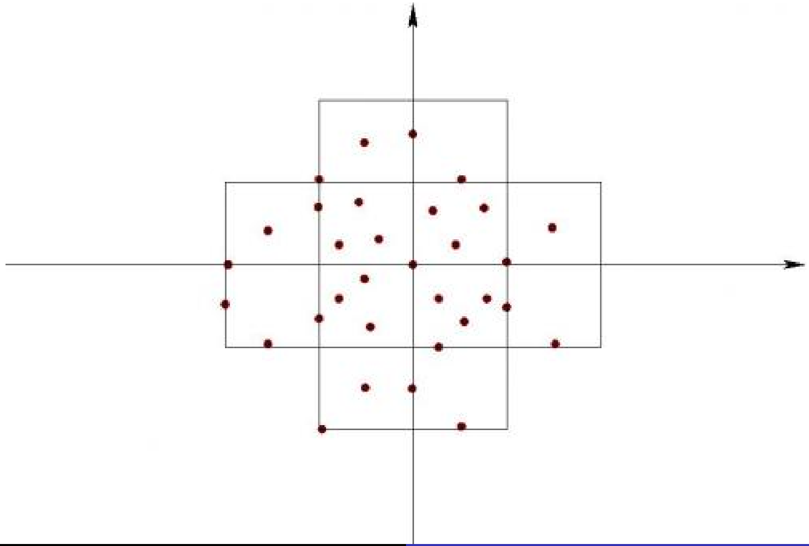
$$x_{n,i} = z_{\eta_{n,i},i}.$$

Use quadrature rule

$$Q(f) = \sum_{n=0}^{2^m-1} \lambda_n f(\boldsymbol{x}_n),$$

where λ_n is chosen equally in each elementary interval such that constant functions are integrated exactly.

Quadrature points in two dimensions



Local structure

Theorem

In each elementary interval is a digitally shifted digital (t, n, s)-net.

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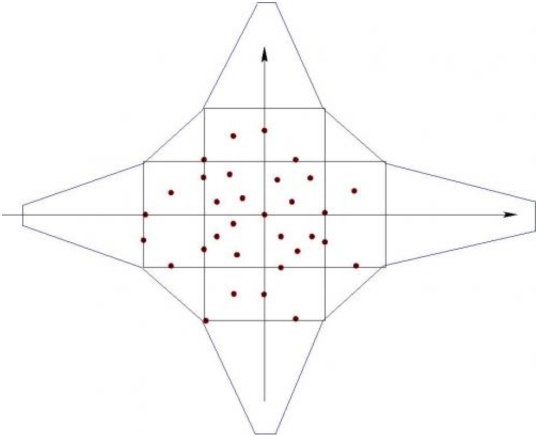
One can think of this as a propagation rule for digital nets.

Global structure (hyperbolic cross shape)



Convergence result

Theorem

Let $f : \mathbb{R}^s \to \mathbb{R}^s$ have mixed partial derivatives of order one in each variable and assume that $\frac{\partial^{|u|} f}{\partial \boldsymbol{x}_u}(\boldsymbol{y})$ goes to 0 at least polynomially (of a certain order) as $|\boldsymbol{y}| \to \infty$. Then

$$\left| \int_{\mathbb{R}^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{n=0}^{N-1} \lambda_n f(\boldsymbol{x}_n) \right| \le C_s \frac{(\log N)^s}{N}$$

Thank You!