On Hammersley and Zaremba point sets in two dimensions

Henri Faure

Institut de mathématiques de Luminy, Marseille, France

faure@iml.univ-mrs.fr

In collaboration with

F. Pillichshammer and G. Pirsic and W. Schmid.

MCQMC2010, Warsaw, August 15–20, 2010

Foreword

This talk is essentialy a survey of recent results (some still in progress) on the L_2 discrepancy of special classes of Hammersley and Zaremba two-dimensional point sets obtained by means of various scramblings using shifts and swapping (reflecting) permutations.

If time permits, we will also give results on the L_p discrepancy. We leave out the star discrepancy studied in a former paper (referred to at the end in a short list of references).

We begin with classic (and necessary) definitions before to give an outline of the talk.

L_p discrepancy

For a point set $\mathcal{P} = \{X_1, X_2, ..., X_N\}$ of N points in $[0,1]^2$ and a subinterval $[0,x) \times [0,y)$, we define the discrepancy function as

$$E(x,y,\mathcal{P}) = A([0,x) \times [0,y), N) - Nxy \quad \text{where}$$

$$A([0,x) \times [0,y), N) = \#\{n; 1 \leq n \leq N, X_n \in [0,x) \times [0,y)\}.$$

Then, the L_p discrepancy is defined as

$$L_p(\mathcal{P}) = \left(\int_0^1 \int_0^1 |E(x,y,\mathcal{P})|^p \, dxdy \right)^{\frac{1}{p}}.$$

A famous theorem of Roth (for $p = 2$, 1954) extended by Schmidt ($1 < p < 2$, 1977) states that (for two dimensions)

$$L_p(\mathcal{P}) \geq c_p \sqrt{\log N}.$$

Sequences in $[0,1)$ and point sets in $[0,1)^2$

Let $b \geq 2$ an integer and $\mathfrak{S}_b$ the set of permutations of $\{0,\ldots,b-1\}$. For any integer $N \geq 1$, the one-dimensional generalized van der Corput sequence S_b^Σ in base b associated with $\Sigma = (\sigma_r)_{r\geq 0}$, $\sigma_r \in \mathfrak{S}_b$, is defined by (radical inverse function)

$$S_b^\Sigma(N) = \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}, \text{ with } N-1 = \sum_{r=0}^{\infty} a_r(N)\ b^r$$

being the b -adic expansion of $N-1$.

If $\Sigma = (\sigma_r) = (\sigma)$ is constant, we write $S_b^\Sigma = S_b^\sigma$.
The van der Corput sequence in base b, S_b^{id}, is obtained with $\sigma = \mathrm{id}$, the identity in $\mathfrak{S}_b$.
The original van der Corput sequence (1935) is S_2^{id}.

Generalized two-dimensional Hammersley points sets $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$.

They are associated with the preceding sequences by taking a finite set $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ instead of a sequence $\boldsymbol{\Sigma}$: For any $n \geq 1$,

$$\mathcal{H}_{b,n}^{\boldsymbol{\sigma}} = \left\{ \left(\sum_{r=0}^{n} \frac{\sigma_r\Big(a_r(N)\Big)}{b^{r+1}}, \frac{N-1}{b^n} \right); 1 \leq N \leq b^n \right\}.$$

If $\boldsymbol{\sigma} = (\sigma, \ldots, \sigma)$, we note $\mathcal{H}_{b,n}^{\sigma} := \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$.

It is known for a long time (Vilenkin, 1967 in base 2 and White, 1975 in base b) that classic Hammersley point sets $\mathcal{H}_{b,n}^{\mathrm{id}}$ don't have optimal order of L_2 discrepancy, which can be stated more precisely as

$$\lim_{n \to \infty} \frac{L_2(\mathcal{H}_{b,n}^{\mathrm{id}})}{\log b^n} = \frac{b^2-1}{12b \log b}.$$

Generalized two-dimensional Zaremba points sets $\mathcal{Z}_{b,n}^{\boldsymbol{\rho}}$.

Generalized Zaremba point sets form a sub-class of generalized Hammersley point sets: Given $\sigma \in \mathfrak{S}_b$, define special permutations σ_l by $\sigma_l(k) := \sigma(k) + l \pmod{b}$ for $0 \leq l, k < b$, called digital shifts.

Then a generalized Zaremba point set $\mathcal{Z}_{b,n}^{\boldsymbol{\rho}}$ is a generalized Hammersley point set associated with

$$\boldsymbol{\rho} = (\rho_0, \ldots, \rho_{n-1}) \in \{\sigma_l; 0 \leq l < b\}^n.$$

Halton and Zaremba (1969) were the first to consider such sets in base 2 with the sequence $\boldsymbol{\rho} = (\mathrm{id}_0, \mathrm{id}_1, \mathrm{id}_0, \mathrm{id}_1, \ldots)$.

Then White (1975) extended Halton and Zaremba construction to arbitrary bases with the sequence

$$\boldsymbol{\rho} = (\mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \ldots).$$

The name Zaremba point sets was given by White in his paper from 1975.

Observe that White's construction needs $n \geq b$, i.e. must contain at least b^b points, to be suitable for the exact order of L_2 discrepancy which is very huge even for small bases.

Besides digital shifts in Zaremba point sets, another permutation plays a leading role in our studies on Hammersley point sets: the permutation τ defined by $\tau(k) := b - k - 1 \pmod b$, called swapping (or reflecting) permutation in base b .

The merit of such constructions is to give the exact order of L_2 discrepancy, moreover with exact formulas, as we will see later.

Contents of the talk

The precise knowledge of discrepancy functions for generalized van der Corput sequences S_b^{Σ} permits to obtain exact formulas for the discrepancies of special classes of Hammersley point sets $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ and Zaremba point sets $\mathcal{Z}_{b,n}^{\boldsymbol{\rho}}$. We first need more definitions with the notion of φ functions.

- 1 Functions $\varphi_{b,h}^{\sigma}$ related to a pair (b, σ)
- 2 Swapping an arbitrary permutation
- 3 Shifting linear permutations
- 4 L_p discrepancy with identity
- 5 Concluding remarks

1. Functions $\varphi_{b,h}^{\sigma}$ related to a pair (b,σ)

These functions are linearizations of one-dimensional discrepancy functions. They are the fundamental tool for the study of irregularities of distribution of sequences S_b^{Σ} and point sets $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$.

Let $\sigma \in \mathfrak{S}_b$ and set $Z_b^{\sigma} = \left(\frac{\sigma(0)}{b}, \cdots, \frac{\sigma(b-1)}{b}\right)$. For any integer h $(0 \leq h \leq b-1)$, we define $\varphi_{b,h}^{\sigma}$ as follows:

Let k be an integer with $1 \leq k \leq b$; then for every $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$ set:

$$\varphi_{b,h}^{\sigma}(x) = A\left(\left[0, \frac{h}{b}\right); k; Z_b^{\sigma}\right) - hx \text{ if } 0 \leq h \leq \sigma(k-1) \quad \text{and}$$
$$\varphi_{b,h}^{\sigma}(x) = (b-h)x - A\left(\left[\frac{h}{b}, 1\right); k; Z_b^{\sigma}\right) \text{ if } \sigma(k-1) < h < b.$$

Finally $\varphi_{b,h}^{\sigma}$ is extended to $\mathbb{R}$ by periodicity. Note that $\varphi_{b,0}^{\sigma} = 0$.

$\varphi_{b,h}^{\sigma}$ functions give rise to other functions, depending only on (b,σ), according to the notion of discrepancy we are dealing with:

For L_2 discrepancy, we need

$$\varphi_b^{\sigma} := \sum_{h=0}^{b-1} \varphi_{b,h}^{\sigma} \quad \text{and} \quad \varphi_b^{\sigma,(2)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^{\sigma})^2.$$

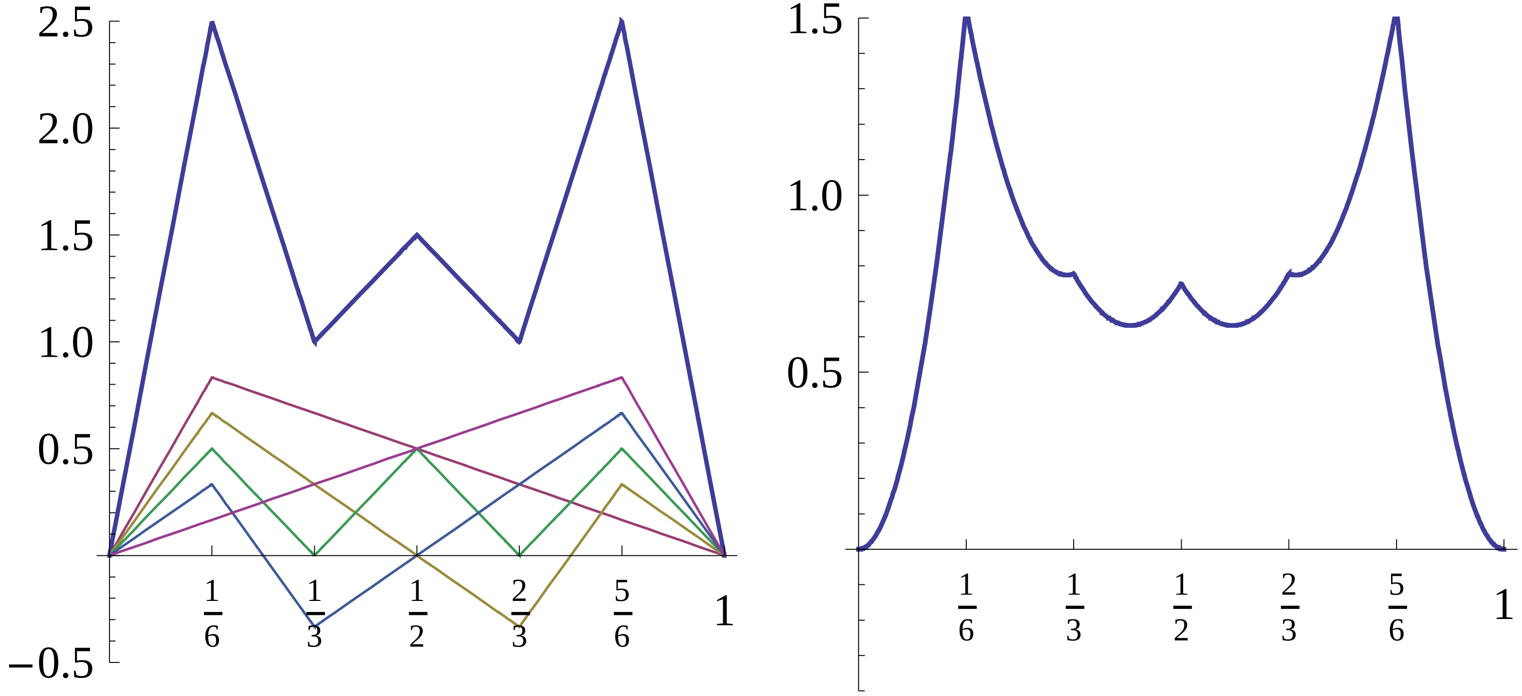
For extreme discrepancy, we need

$$\psi_b^{\sigma,+} := \max_{0 \le h \le b-1} (\varphi_{b,h}^{\sigma}) \text{ and } \psi_b^{\sigma,-} := \max_{0 \le h \le b-1} (-\varphi_{b,h}^{\sigma}).$$

Here is the origin of the term "swapping" because we first used τ to improve the extreme discrepancy of S_b^{Σ} and $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$:

Let $\overline{\sigma} := \tau \circ \sigma$. Then $\psi_b^{\overline{\sigma},+} = \psi_b^{\sigma,-}$ and $\psi_b^{\overline{\sigma},-} = \psi_b^{\sigma,+}$, i.e, τ "swaps" $\psi_b^{\sigma,+}$ for $\psi_b^{\sigma,-}$ and $\psi_b^{\sigma,-}$ for $\psi_b^{\sigma,+}$.

As for the L_2 discrepancy, we have $\varphi_b^{\overline{\sigma}} = -\varphi_b^{\sigma}$ and $\varphi_b^{\overline{\sigma},(2)} = \varphi_b^{\sigma,(2)}$ and we can say that τ "swaps" the signs of φ_b^{σ} and $\varphi_b^{\overline{\sigma}}$.



The functions $\varphi_{b,h}^{\sigma}$, $0 \leq h < b$ and φ_b^{σ} (left) and $\varphi_b^{\sigma,(2)}$ (right) for $b = 6$ and $\sigma = (4, 1)$.

Interlude

Recall that Hammersley and Zaremba two-dimensional points sets are in the form

$$\mathcal{H}_{b,n}^{\boldsymbol{\sigma}} = \left\{ \left(\sum_{r=0}^{n} \frac{\sigma_r\Big(a_r(N)\Big)}{b^{r+1}}, \frac{N-1}{b^n} \right) ; 1 \leq N \leq b^n \right\}.$$

with $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$, $\sigma_r \in \mathfrak{S}_b$.

In the following, we are concerned with two sub-classes: for a given permutation $\sigma \in \mathfrak{S}_b$, we consider finite sequences

(1) $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\sigma, \tau \circ \sigma\}^n$ (swapping)

(2) $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_{n-1}) \in \{\sigma_l \;;\; 0 \leq l < b\}^n$ (shifting).

In base 2, both cases coincide with $\sigma = \mathrm{id}$ (Halton-Zaremba paper).

2. Swapping an arbitrary permutation

Theorem 1 *Let* $\boldsymbol{\sigma} \in \{\sigma, \overline{\sigma}\}^n$*, with* $\overline{\sigma} = \tau \circ \sigma$*, and let* l *denote the number of components of* $\boldsymbol{\sigma}$ *which are equal to* σ*. Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = (\Phi_b^\sigma)^2((n-2l)^2 - n) + O(n),$$

where $\Phi_b^\sigma := \frac{1}{b}\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x$ *and where the constant in the* O *notation only depends on* b*.*

Theorem 1 shows that one can always obtain $L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = O(\sqrt{n})$ which is the best possible with respect to Roth's lower bound: either take σ such that $\Phi_b^\sigma = 0$ or, for arbitrary σ, take l such that the term $(n-2l)^2 = O(n)$. But this gives no information on the leading term.

A weak technical condition on σ permits to get an exact formula and hence to study the constant in the leading term of Theorem 1:

Theorem 2 *Let σ such that $\tau \circ \sigma = \sigma \circ \tau$ and let $\overline{\sigma} = \tau \circ \sigma$. For $\boldsymbol{\sigma} \in \{\sigma, \overline{\sigma}\}^n$, let l denote the number of components of $\boldsymbol{\sigma}$ which are equal to σ. Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = (\Phi_b^\sigma)^2((n-2l)^2 - n) + \Phi_b^\sigma \left(1 - \frac{1}{2b^n}\right)(2l-n) + n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}},$$

where $\Phi_b^\sigma := \frac{1}{b}\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x$ and $\Phi_b^{\sigma,(2)} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x$.

Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ only depends on n, b, σ and l. It does not depend on the distribution of σ and $\overline{\sigma}$ in $\boldsymbol{\sigma}$.

From Theorem 2, if $\Phi_b^\sigma \neq 0$, one can choose l such that $(n-2l)^2 = O(1)$ to obtain the best possible order of L_2 discrepancy in the sense of Roth's lower bound, the simplest choice being $l = \frac{n}{2}$ if n is even and $l = \frac{n-1}{2}$ if n is odd.

Corollary 1 *Let σ such that $\tau \circ \sigma = \sigma \circ \tau$, with $\Phi_b^\sigma \neq 0$. Then we have*

$$\min_{\boldsymbol{\sigma} \in \{\sigma, \overline{\sigma}\}^n} \left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \right)^2 = n \left(\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2 \right) + O(1). \textit{ Especially}$$

$$\lim_{n \to \infty} \min_{\boldsymbol{\sigma} \in \{\sigma, \overline{\sigma}\}^n} \frac{L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})}{\sqrt{\log b^n}} = \min_{\{\sigma : \tau \circ \sigma = \sigma \circ \tau\}} \sqrt{\frac{\Phi_b^{\sigma,(2)} - \left(\Phi_b^\sigma\right)^2}{\log b}}.$$

With this formula, we can search for permutations σ which yield the best result. Full search was performed by Pirsic and Schmid (coauthors of this section) for $4 \leq b \leq 23$.

b	$\frac{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2}{\log b}$	num. value	b	$\frac{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2}{\log b}$	num. value
2	$\frac{5}{192 \log(2)}$	0.037570	13	$\frac{574}{6591 \log(13)}$	0.033953
3	$\frac{4}{81 \log(3)}$	0.044950	14	$\frac{41581}{460992 \log(14)}$	0.034178
4	$\frac{5}{96 \log(4)}$	0.037570	15	$\frac{4714}{50625 \log(15)}$	0.034385
5	$\frac{112}{1875 \log(5)}$	0.037114	16	$\frac{17573}{196608 \log(16)}$	0.032237
6	$\frac{343}{5184 \log(6)}$	0.036927	17	$\frac{8040}{83521 \log(17)}$	0.033977
7	$\frac{512}{7203 \log(7)}$	0.036529	18	$\frac{40631}{419904 \log(18)}$	0.033478
8	$\frac{5}{64 \log(8)}$	0.037570	19	$\frac{12970}{130321 \log(19)}$	0.033800
9	$\frac{512}{6561 \log(9)}$	0.035516	20	$\frac{46733}{480000 \log(20)}$	0.032500
10	$\frac{3391}{40000 \log(10)}$	0.036817	21	$\frac{19402}{194481 \log(21)}$	0.032768
11	$\frac{3680}{43923 \log(11)}$	0.034940	22	$\frac{278629}{2811072 \log(22)}$	**0.032066**
12	$\frac{1759}{20736 \log(12)}$	0.034137	23	$\frac{87112}{839523 \log(23)}$	0.033093

If $\Phi_b^\sigma = 0$, the formula from Theorem 4 is independent of l (and we can take $l = n$). Hence, only one permutation can give the exact order of L_2 discrepancy, but we find that some bases are forbiden.

Corollary 2 *Let σ such that $\tau \circ \sigma = \sigma \circ \tau$, with $\Phi_b^\sigma = 0$. Then for any $\boldsymbol{\sigma} \in \{\sigma, \overline{\sigma}\}^n$ we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = \left(L_2(\mathcal{H}_{b,n}^{\sigma})\right)^2 = n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.$$

It is easy to compute Φ_b^σ (and hence to get (b, σ) with $\Phi_b^\sigma = 0$):

$$\Phi_b^\sigma = \frac{1}{b}\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x = \frac{1}{b^2}\sum_{k=0}^{b-1} \sigma(k)k - \left(\frac{b-1}{2}\right)^2.$$

Hence $\Phi_b^\sigma = 0$ is impossible for bases $b = 3, 7$, and $b \equiv 2 \pmod 4$. For other bases, we have explicit constructions and many permutations by computer search (for $b \leq 17$).

3. Shifting linear permutations

The study of shifts of Hammersley point sets requires formulas for φ functions associated to shifts of a given permutation σ. In case of identity, it is easy to show that

$$\forall x \in \mathbb{R} \quad \varphi_b^{\mathrm{id}_l}(x) = \varphi_b^{\mathrm{id}}\left(x + \frac{l}{b}\right) - \varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right).$$

But for an arbitrary permutation, the relations are much more difficult to bring to the fore and are complex a lot. Here is the main reason of our choice –in a first step– of linear permutations for which we get:

$$\forall x \in \mathbb{R} \quad \varphi_b^{\pi_l}(x) = \varphi_b^{\pi}\left(x + \frac{\pi^{-1}(l)}{b}\right) - \varphi_b^{\pi}\left(\frac{\pi^{-1}(l)}{b}\right).$$

Thanks to this formula, we obtain the L_2 discrepancy of Zaremba point sets $\mathcal{Z}_{b,n}^{\boldsymbol{\sigma}}$ associated with linear permutations:

Theorem 3

Let $\pi \in \mathfrak{S}_b$ be linear and let $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\pi_l \,;\, 0 \le l < b\}^n$. For $0 \le l < b$ define $\lambda_l := \#\{0 \le i < n : \sigma_i = \pi_l\}$ and $l' = \pi^{-1}(l)$. Then we have

$$
\begin{aligned}
\left(L_2(\mathcal{Z}_{b,n}^{\boldsymbol{\sigma}})\right)^2 &= \left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(b\Phi_b^{\pi} - \varphi_b^{\pi}\left(\frac{l'}{b}\right)\right)\right)^2 + n(\Phi_b^{\pi,(2)} + \Phi_b^{\pi}) \\
&+ \sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left[\varphi_b^{\pi,(2)}\left(\frac{l'}{b}\right) - 2F_b^{\pi}(l) - \frac{1}{b}\left(b\Phi_b^{\pi} - \varphi_b^{\pi}\left(\frac{l'}{b}\right)\right)^2\right] \\
&- \frac{1}{2b}\sum_{l=0}^{b-1} \lambda_l \left(\varphi_b^{\pi}\left(\frac{l'}{b}\right) + \varphi_b^{\pi^{-1}}\left(\frac{b-l}{b}\right)\right) + O(1), \text{ where}
\end{aligned}
$$

$$F_b^{\pi}(l) := \frac{1}{b}\sum_{h,j=0}^{b-1} \varphi_{b,h}^{\pi}\left(\frac{\pi^{-1}(l)}{b}\right)\varphi_{b,h}^{\pi}\left(\frac{j}{b}\right) \quad (0 \le l < b).$$

Corollary 3 If $\lambda_l = \lfloor n/b \rfloor + \theta_l$ with $\theta_l \in \{0,1\}$, then we have

$$\begin{aligned}\left(L_2(\mathcal{Z}_{b,n}^{\boldsymbol{\sigma}})\right)^2 \; = \; & n\left(2\Phi_b^{\pi,(2)} - (\Phi_b^{\pi})^2 + 2(\Phi_b^{\pi})^2 + \frac{b^2-1}{36b^2}\right) \\ & -n\left(\frac{2}{b^2}\sum_{l=0}^{b-1} F_b^{\pi}(l) - \frac{1}{b^3}\sum_{l=0}^{b-1}\left(\varphi_b^{\pi}\left(\frac{l}{b}\right)\right)^2\right) + O(1).\end{aligned}$$

The leading constant is coarse and requires further simplifications. On the basis of prime bases less than 100, thanks to F. Pausinger for computer checking, we conjecture that it reduces to $n(\Phi_b^{\pi,(2)} - (\Phi_b^{\pi})^2)$, like in the study with the swapping permutation τ.

In the special case of identity, we recover preceding results (in FP mcqmc08 and JTN Bordeaux, submitted):

Theorem 4 *Let* $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\mathrm{id_l} : 0 \le l < b\}^n$, *then*

$$\left(L_2(\mathcal{Z}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = \left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right)^2 + n\frac{(b^2-1)(3b^2+13)}{720b^2}$$

$$\left(1 - \frac{1}{2b^n}\right)\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) + O(1),$$

where for $0 \le l < b \lambda_l := \#\{0 \le i < n : \sigma_i = \mathrm{id}_l\}$.

If we choose $\lambda_0 = n$ and $\lambda_1 = \ldots = \lambda_{b-1} = 0$, then $\mathcal{Z}_{b,n}^{\boldsymbol{\sigma}} = \mathcal{H}_{b,n}$, the classical Hammersley point set.
If we choose $\lambda_l = n$ for some $l \in \{0, \ldots, b-1\}$ and $\lambda_i = 0$ for all $i \neq l$, then we obtain Theorem 1 from mcqmc08.

Corollary 4 *Let* $\boldsymbol{\sigma} \in \{\mathrm{id}_l : 0 \le l < b\}^n$ *such that* $\lambda_l = \left\lfloor \frac{n}{b} \right\rfloor + \theta_l$ *with* $\theta_l \in \{0, 1\}$ *for all* $0 \le l < b$. *Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1).$$

This generalizes the result of White. But Theorem 4 permits more: only one non-zero shift is enough to get the same result. Suppose $b = 2c+1$ is odd (a similar result holds for b even too).

Corollary 5 *With* $\boldsymbol{\sigma} \in \{\mathrm{id}_l : 0 \le l < b\}^n$ *such that*

$$\lambda_0 = \left\lceil \frac{n}{3} \right\rceil, \lambda_c = \left\lfloor \frac{2n}{3} \right\rfloor \text{ and } \lambda_l = 0 \text{ for } l \notin \{0, c\}, \text{ we have}$$

$$\left(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1).$$

This corollary can be extended to linear permutations.

4. L_p discrepancy with identity

With identity, the discrepancy function is always positive and this permits to obtain results with the L_p discrepancy. To end this communication, we give three results obtained with FP.

Theorem 5 *Let $p \in \mathbb{N}$. For any $n \geq 1$ we have*

$$(L_p(\mathcal{H}_{b,n}^{\text{id}}))^p = n^p \left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}) \text{ and hence}$$

$$\lim_{n\to\infty} \frac{L_p(\mathcal{H}_{b,n}^{\text{id}})}{\log b^n} = \frac{b^2-1}{12b\log b}.$$

Theorem 5 shows that $L_p(\mathcal{H}_{b,n}^{\text{id}})$ is asymptotically not of best possible order with respect to the lower bound of Roth and Schmidt ($\sqrt{n}$).

This disadvantage of $\mathcal{H}_{b,n}^{\text{id}}$ can be overcome thanks to τ:

Theorem 6 *Let p be an even positive integer. Then for any $n \geq 1$ and $\boldsymbol{\sigma} \in \{id, \tau\}^n$, we have*

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \{id,\tau\}^n} \left(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p \leq 2 \left(\frac{b^2-1}{12}\right)^p \frac{p!}{(p/2)!\, 2^{p/2}} n^{p/2} + O(n^{p/2-1}).$$

Corollary 6 *For even p, there exists a sequence $\boldsymbol{\sigma}^* \in \{id, \tau\}^n$ such that*

$$(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}))^p \leq 2 \left(\frac{b^2-1}{12}\right)^p \frac{p!}{(p/2)!\, 2^{p/2}} n^{p/2} + O(n^{p/2-1}).$$

For $b = 2$ this is an improvement of a result of Kritzer and Pillichshammer.

5. Concluding remarks

• So far, we have two ways to get the exact order (with leading terms) of two-dimensional L_2 discrepancy with modifications of two-dimensional Hammersley point sets:

(1) Using shifts works for $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ with
$\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\pi_l\,;\, 0 \leq l < b\}^n$ when π is a linear permutation.

(2) Using the swapping permutation τ works for $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ with
$\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\sigma, \tau \circ \sigma\}^n$ when σ satisfies $\tau \circ \sigma = \sigma \circ \tau$.

In base 2, both methods coincide with $\sigma = \pi = \mathrm{id}$.

• It should be fine to remove both restrictive conditions above in the two ways we have explored.

• Last but not least, a recent result from A. Hinrichs and L. Markashin greatly improves the lower bound of Roth, so that both upper and lower constants are now close enough: $0.038925\ldots < c_2 < 0.17907\ldots$.

The talk was based on the following articles and manuscripts:

- H. Faure: Star extreme discrepancy of generalized two-dimensional Hammersley point sets. UDT 3, no.2: 45–65, 2008.

- H. Faure and F. Pillichshammer: L_p discrepancy of generalized two-dimensional Hammersley point sets. Monatsh.M. 158: 31–61, 2009.

- H. Faure and F. Pillichshammer: L_2 discrepancy of two-dimensional digitally shifted Hammersley point sets in base b. MCQMC2008, P. L'Ecuyer and A. Owen (eds.), Springer: 355–368, 2009.

- H. Faure, F. Pillichshammer, G. Pirsic and W. Ch. Schmid: L_2 discrepancy of generalized two-dimensional Hammersley point sets scrambled with arbitrary permutations. Acta Arith. 141.4: 395–418, 2010.

- H. Faure and F. Pillichshammer: L_2 discrepancy of generalized Zaremba point sets (I) submitted and (II) in progress.