Discrete time hedging in the Lévy model

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MCQMC Warsaw 2010

1. The setting

- $S_t = e^{X_t}$, $(X_t)_{t\geq 0}$ Lévy process
- Lévy-Itô decomposition

$$X_t = \gamma t + \sigma W_t + \int_{(0,t]\times\{1<|x|\}} x dN(s,x) + \int_{(0,t]\times\{0<|x|\leq 1\}} x d\tilde{N}(s,x),$$

- ν Lévy measure,
- $(S_t)_{t\geq 0}$ L_2- martingale $\iff$

$$\int_{\{|x|\geq 1\}} e^{2x}\nu(dx) < \infty,$$

and

$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{\{|x|\leq 1\}} x)\nu(dx) = 0.$$

- $g : (0,\infty) \to \mathbb{R}$ Borel function, $g(S_1) \in L_2$.
- Example: $g = \mathbb{I}_{[K,\infty)}$, $K > 0$.

Galtchouk-Kunita-Watanabe decomposition

$$g(S_1) = \mathbb{E}g(S_1) + \int_0^1 \phi_t^g \, dS_t + \mathcal{N}$$

- $\mathcal{N}$ is orthogonal to all stochastic integrals with respect to (S_t),
- $\mathcal{N} \neq 0$ in general,
- (ϕ_t^g) predictable process.

Let

$$\Pi : g(S_1) \mapsto \int_0^1 \phi_t^g \, dS_t$$

be the orthogonal projection onto the stochastic integral.

2. Discrete time approximation

Let $\tau_N := \{0 = t_0 < t_1 < \ldots < t_N = 1\}$.

$$\left\| \int_0^1 \phi_t^g \, dS_t - \sum_{k=1}^{N} \phi_{t_{k-1}}^g (S_{t_k} - S_{t_{k-1}}) \right\|_{L_2} \leq c N^{-r}$$

- best possible r ?
- r depends on the pattern of the (τ_N) and on the Malliavin fractional smoothness of $\int_0^1 \phi_t^g \, dS_t$.

3. Malliavin fractional smoothness

Chaos expansion

Let

$$d\mathrm{m}(t,x) := [\sigma^2 d\delta_0(x) + x^2 d\nu(x)]dt = d\mu(x)dt.$$

Define the random measure generated by X on $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mathrm{m}(B) < \infty$,

$$M(B) \quad := \quad \sigma \int_{\{t\in\mathbb{R}_+:(t,0)\in B\}} dW_t + \lim_{n\to\infty} \int_{\{(t,x)\in B:1/n<|x|<n\}} x\ \tilde{N}(dt,dx).$$

There exists a **chaos expansion**

$$F = \sum_{m=0}^{\infty} I_m\left(f_m\right),\ a.s. \quad F \in L_2(\Omega, \mathcal{F}^X, \mathbb{P}).$$

- $I_m(f_m)$ is the **multiple integral** w.r.t. M: $I_0(f_0) = \mathbb{E}F$,
  $I_1(f_1) = \int_{\mathbb{R}_+\times\mathbb{R}} f_1(t,x)M(dt,dx)$,
  $I_2(f_2) = 2\int_{\mathbb{R}_+\times\mathbb{R}}\int_{(0,t)\times\mathbb{R}} \tilde{f}_2((s,y),(t,x))M(ds,dy)M(dt,dx)$.

- $\mathbb{D}_{1,2}$ is the space of all $F \in L_2$ such that

$$\sum_{m=0}^{\infty}(1+m)\| \, I_m(f_m) \, \|_{L_2}^2 < \infty,$$

where $\| \, I_m(f_m) \, \|_{L_2}^2 = m!\|\tilde{f}_m\|_{L_2^m}^2$ and

$$L_m^2 := L_2((R_+\times\mathbb{R})^m, \mathcal{B}(R_+\times\mathbb{R})^{\otimes m}, \mathrm{m}^{\otimes m}).$$

- **Malliavin derivative** $\mathrm{m}\otimes\mathbb{P}$ a.s.

$$D_{t,x}F := \sum_{m=0}^{\infty} mI_{m-1}\left(\, \tilde{f}_m((t,x),\cdot)\,\right), \quad F \in \mathbb{D}_{1,2}.$$

Definition 3.1

for $q \in [1, \infty]$ and $\theta \in (0,1)$ we define

$$B^{\theta}_{2,q} := (L_2, \mathbb{D}_{1,2})_{\theta,q}$$

by real interpolation.

It holds

- $B^{\theta}_{2,2}$ is the space of all $F \in L_2$ such that

$$\|F\|^2_{B^{\theta}_{2,2}} = \sum_{m=0}^{\infty} (1 + m^{\theta}) \| \; I_m(f_m) \, \|^2_{L_2} < \infty,$$

  we set $B^{1}_{2,2} := \mathbb{D}_{1,2}$,
- for $0 < \theta_1 < \theta_2 < 1$ and $q_1, q_2 \in [1, \infty]$

$$\mathbb{D}_{1,2} \subseteq B^{\theta_2}_{2,\min\{q_1,q_2\}} \subseteq B^{\theta_2}_{2,q_2} \subseteq B^{\theta_1}_{2,q_1} \subseteq L_2.$$

4. Special (τ_N) imply the best rate $r = \frac{1}{2}$

$$a(g(S_1);\tau_N) := \left\| \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (\phi_t^g - \phi_{t_{k-1}}^g) dS_t \right\|_{L_2}$$

$$\tau_N^\theta := \left\{ t_k := 1 - \left(1 - \frac{k}{N}\right)^{\frac{1}{\theta}}, k = 0, \dots, N \right\}.$$

For $g(S_1) = \sum_{m=0}^{\infty} I_m \left(g_m \, \mathbb{1}_{[0,1]}^{\otimes m} \right)$ we have

$$\phi_t^g = \frac{1}{c_\mu S_{t-}} \sum_{m=1}^{\infty} m I_{m-1} \left(\int_{\mathbb{R}} \tilde{g_m}(x,\cdot) q(x) \mu(dx) \, \mathbb{1}_{[0,t)}^{\otimes (m-1)} \right) \mathbb{1}_{[0,1]}(t),$$

where $q(x) = \mathbb{1}_{\{0\}}(x) + \frac{e^x - 1}{x} \mathbb{1}_{\mathbb{R}_0}(x)$ and $c_\mu = \int q(x)^2 \mu(dx)$.

By a martingale representation theorem

$$S_{t-}\phi_t^g = c + \int_{(0,t)\times\mathbb{R}} \kappa^g(s,x) M(ds,dx),$$

$$\kappa^g(s,x) \quad = \quad \sum_{m=2}^{\infty} \frac{m(m-1)}{c_\mu} I_{m-2}\left(\int_{\mathbb{R}} \tilde{g_m}(y,x,\cdot) q(y)\mu(dy) \ \mathbb{1}_{[0,s)}^{\otimes(m-2)} \right).$$

$$H(t) := \left(\mathbb{E} \int_{\mathbb{R}} |\kappa^g(t,x) - S_{t-}\phi_t^g q(x)|^2 \mu(dx) \right)^{\frac{1}{2}}.$$

Theorem 4.1

Let $0 < \theta \leq 1$. *TFAE*

1. $\Pi(g(S_1)) \in B^{\theta}_{2,2}$.
2. $\int_0^1 (1-t)^{1-\theta} H(t)^2 dt < \infty$.
3. *There exists a* $c > 0$ *such that for* $N = 1, 2, \ldots$

$$a(g(S_1); \tau_N^{\theta}) \leq \frac{c}{\sqrt{N}}.$$

Moreover, if one of the assertions is fulfilled
$\lim_{N\to\infty} N\, a(g(S_1); \tau_N^{\theta})^2 = c_\mu \int_0^1 (1-t)^{1-\theta} H(t)^2 dt.$

Theorem 4.2

If $g(S_1)$ is not a.s. a multiple of S_1 then

$$\liminf_{N\to\infty} \sqrt{N}\, a(g(S_1);\tau_N) > 0.$$

5. Fractional smoothness of $\mathbb{1}_{[K,\infty)}(S_1)$

(γ, σ^2, ν)	$\mathbb{1}_{[K,\infty)}(S_1)$
$\sigma > 0$ $\int e^{2x}\nu(dx) < \infty$	$B^{\frac{1}{2}}_{2,\infty}$ $\implies B^{\theta}_{2,2} \quad \forall\theta < \frac{1}{2}$
Compound Poisson	$\mathbb{D}_{1,2}$
$(0, 0, \nu)$ Cauchy process: $d\nu(x) = \|x\|^{-2}dx$	$\mathbb{1}_{[K,\infty)}(S_1) \in B^{\theta}_{2,2}, \theta \in (0,1)$ and $\mathbb{1}_{[K,\infty)}(S_1) \notin \mathbb{D}_{1,2}$

stable like behavior for small jumps, $\sigma = 0$	$\Pi(\mathbb{1}_{[K,\infty)}(S_1))$
with Blumenthal-Getoor index $\alpha \in (0, \frac{3}{2})$	$\mathbb{D}_{1,2}$
$\alpha \in (\frac{3}{2}, 2)$	$B_{2,\infty}^{\frac{3}{\alpha}-1}$

$$\nu(dx) = \frac{f(x)}{|x|^{1+\alpha}}dx, \quad \lim_{x\downarrow 0} f(x) = f_+ > 0, \quad \lim_{x\uparrow 0} f(x) = f_- > 0$$

6. Equidistant time nets

Theorem 6.1

Let $q \in [0,\infty]$ *and* $\theta \in (0,1)$. *Then*

$$\|\Pi(g(S_1))\|_{B^{\theta}_{2,q}} \sim_c \left\| \int_0^1 \phi^g_t \, dS_t \right\|_{L_2} + \left\| \left(N^{\frac{\theta}{2}-\frac{1}{q}} a(g(S_1); \tau^{\mathbf{1}}_N) \right)_{N=1}^{\infty} \right\|_{l_q}.$$

In particular it holds

$$\Pi(g(S_1)) \in B^{\theta}_{2,\infty} \iff \text{there exists } c>0 \text{ such that } \forall N = 1,2,\ldots \quad a(g(S_1); \tau^{\mathbf{1}}_N)) \le c N^{-\frac{\theta}{2}}.$$

7. References

- M. Brodén and P. Tankov
  Tracking errors from discrete hedging in exponential Lévy models *arXiv:1003.0709v1 (2010).*
- S. Geiss
  Quantitative approximation of certain stochastic integrals
  Stochastics and Stochastic Reports, 76(2004)339-362.
- S. Geiss and M. Hujo.
  Interpolation and approximation in $L_2(\gamma)$.
  Journal of Approximation Theory 144(2007)213-232.