Fractional Discrepancy

Michael Gnewuch

Department of Computer Science
Columbia University, New York

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Joint Work with
Josef Dick (UNSW, Sydney)

Fractional Function Spaces ($s = 1$)

Let $\alpha \in (0, 1]$, $p > \alpha^{-1}$ (iff $p'(1-\alpha) < 1$).

$$\mathcal{H}_{\alpha,p} := \left\{ f \,\middle|\, \exists \tilde{f} \in L_p, \exists c \in \mathbb{R} : f = c + \frac{1}{\Gamma(\alpha)} \int_0^x \tilde{f}(t)(\cdot - t)^{\alpha-1} \,\mathrm{d}t \right\}.$$

$f \in \mathcal{H}_{\alpha,p}$ Hölder continuous with exponent $\alpha - 1/p$ and $f(0) = c$.

Fractional derivative of order $\alpha \in (0, 1)$ (Riemann-Liouville)

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t)(x-t)^{-\alpha} \,\mathrm{d}t, \tag{1}$$

Modified differential operator

$$D^\alpha f(t) = \frac{\mathrm{d}^\alpha (f(t) - f(0))}{\mathrm{d}t^\alpha}.$$

Then $\tilde{f} = D^\alpha f$.

Fractional Function Spaces ($s \geq 1$)

$S := \{1, 2, \ldots, s\}$.

$$\mathcal{H}_{\alpha,s,p} := \{f \mid \forall u \subseteq S \, \exists \tilde{f}_u \in L_p([0,1]^{|u|}) : f = \Phi((\tilde{f}_u)_{u \subseteq S})\},$$

where

$$\Phi((\tilde{f}_u)_{u\subseteq S})(\mathbf{x}) := \sum_{u \subseteq S} \Gamma(\alpha)^{-|u|} \int_{[0,1]^{|u|}} \tilde{f}_u(\mathbf{t}_u) \prod_{j \in u} (x_j - t_j)_+^{\alpha-1} \, \mathrm{d}\mathbf{t}_u.$$

$$\|f\|_{\alpha,s,p} := \left(\sum_{u \subseteq S} \Gamma(\alpha)^{-p|u|} \|\tilde{f}_u\|^p_{L_p([0,1]^{|u|})} \right)^{1/p}$$

Fractional Koksma Hlawka Inequality

$X := \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\} \subset [0,1]^s$.

$$\Delta_\alpha(\mathbf{t}_u, u, X) = \alpha^{-|u|} \prod_{j \in u} (1 - t_j)^\alpha - \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j \in u} (x_{n,j} - t_j)_+^{\alpha - 1}.$$

Fractional discrepancy

$$D^*_{\alpha,s,q}(P) = \left(\sum_{\emptyset \neq u \subseteq S} \int_{[0,1]^{|u|}} |\Delta_\alpha(\mathbf{t}_u, u, P)|^q \, \mathrm{d}\mathbf{t}_u \right)^{1/q}$$

Theorem [Dick 08]. Let $s \geq 1$, $0 < \alpha \leq 1$, $p > \alpha^{-1}$. Then for any $f \in \mathcal{H}_{\alpha,s,p}$ we have

$$\left| \int_{[0,1]^s} f(\mathbf{t}) \, \mathrm{d}\mathbf{t} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \leq D^*_{\alpha,s,p'}(X) \|f\|_{\alpha,s,p}.$$

Fractional Koksma Hlawka (In-)Equality is Sharp

$X := \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\} \subset [0,1]^s.$

Worst-case error of quasi-Monte Carlo integration

$$e^{\mathrm{wor}}(X, \mathcal{H}_{\alpha,s,p}) := \sup_{\|f\|_{\alpha,s,p} \le 1} \left| \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right|.$$

Theorem. Let $s \in \mathbb{N}$, $0 < \alpha \le 1$, $p > \alpha^{-1}$. For $X \subset [0,1]^s$ we have

$$e^{\mathrm{wor}}(X, \mathcal{H}_{\alpha,s,p}) = D^*_{\alpha,s,p'}(X).$$

Besov Spaces ($s = 1$)

For $f \in L_p([0,1])$, $1 \le p < \infty$, and $0 < \delta < 1$ define the integral modulus of continuity

$$\omega_p(f,\delta) := \sup_{h \le \delta} \left(\int_0^{1-h} |f(x+h) - f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Besov space for $\alpha \in (0,1)$

$$B^\alpha_{p,q} := \{ f \in L_p \mid \|f\|_{B^\alpha_{p,q}} < \infty \},$$

where

$$\|f\|_{B^\alpha_{p,q}} := \|f\|_{L_p} + \left(\sum_{j=0}^{\infty} b^{j\alpha q} \omega_p(f, b^{-j})^q \right)^{1/q}.$$

Theorem. Let $p \in [1,\infty)$ and $\alpha \in (1/2, 1)$. Then we have the continuous embeddings $B^\alpha_{p,\min\{p,2\}} \hookrightarrow \mathcal{H}_{\alpha,p} \hookrightarrow B^\alpha_{p,\max\{p,2\}}$.
In particular,

$$B^\alpha_{2,2} \simeq \mathcal{H}_{\alpha,2}.$$

Haar Wavelet Spaces ($s = 1$)

$\{\psi_{i,k}^{j}\}_{j,k,i}$ orthonormal Haar wavelet basis of $L_2([0,1])$.

Each $\psi_{i,k}^{j}$ supported on $[b^{-j}k, b^{-j}(k+1))$.

Haar wavelet space

$$\mathcal{H}_{\mathrm{wav},\alpha,p} := \{f \in L_1 \mid \|f\|_{\mathrm{wav},\alpha,p} < \infty\},$$

where

$$\|f\|_{\mathrm{wav},\alpha,p}^{p} := \sum_{j=0}^{\infty} b^{j\alpha p} \sum_{k \in \Delta_{j-1}} \sum_{i \in \nabla_j} |\langle f, \psi_{i,k}^{j} \rangle_{L_2}|^{p}.$$

Embedding the Fractional Space into the Wavelet Space

Theorem. For $\alpha \in (0,1)$ the Besov space $B^{\alpha}_{2,2}$ is continuously embedded in the Haar wavelet space $\mathcal{H}_{\mathrm{wav},\alpha,2}$. More precisely,

$$\|f\|_{\mathrm{wav},\alpha,2} \leq 2\sqrt{1+b^{2\alpha}}\,\|f\|_{B^{\alpha}_{2,2}} \quad \text{for all } f \in B^{\alpha}_{2,2}.$$

Corollary. For $\alpha \in (0,1)$ the fractional space $\mathcal{H}_{\alpha,2}$ is continuously embedded in the Haar wavelet space $\mathcal{H}_{\mathrm{wav},\alpha,2}$, i.e., there exists $C > 0$ such that

$$\|f\|_{\mathrm{wav},\alpha,2} \leq C\,\|f\|_{\alpha,2} \quad \text{for } f \in \mathcal{H}_{\alpha,2}.$$

Haar Wavelet Spaces ($s \geq 1$)

s-dimensional wavelet functions: $\Psi^{\mathbf{j}}_{\mathbf{i},\mathbf{k}} := \otimes_{\ell=1}^{s} \psi^{j_\ell}_{i_\ell,k_\ell}$.

s-variate wavelet space

$$\mathcal{H}_{\mathrm{wav},\alpha,s,p} := \{ f \in L_1([0,1]^s) \mid \|f\|_{\mathrm{wav},\alpha,s,p} < \infty \},$$

where

$$\|f\|^p_{\mathrm{wav},\alpha,s,p} := \sum_{L=0}^{\infty} b^{L\alpha p} \sum_{|\mathbf{j}|_1 = L} \sum_{\mathbf{k} \in \Delta_{\mathbf{j}-\mathbf{1}}} \sum_{\mathbf{i} \in \nabla_{\mathbf{j}}} |\langle f, \Psi^{\mathbf{j}}_{\mathbf{i},\mathbf{k}} \rangle_{L_2}|^p.$$

Corollary. Let $\alpha \in (1/2, 1)$. $\mathcal{H}_{\alpha,s,2}$ is continuously embedded in $\mathcal{H}_{\mathrm{wav},\alpha,s,2}$. In particular, there exists $C > 0$ such that

$$\|f\|_{\mathrm{wav},\alpha,s,2} \leq C^s \|f\|_{\alpha,s,2} \quad \text{for all } f \in \mathcal{H}_{\alpha,s,2},$$

and for each set of points $P \subset [0,1]$

$$e^{\mathrm{wor}}(P, \mathcal{H}_{\alpha,s,2}) \leq C^s \, e^{\mathrm{wor}}(P, \mathcal{H}_{\mathrm{wav},\alpha,s,2}).$$

Quasi-Monte Carlo Integration Points ($s \geq 1$)

X $(t, L+t, s)$-net in base b, $N := |X| = b^{-L-t}$.

$$Q_{s,L}(f) = \frac{1}{N} \sum_{\mathbf{x} \in X} f(x).$$

Theorem. $\alpha > 1/2$, $p \in [1, \infty)$. There exists $C > 0$ such that

$$e^{\mathrm{wor}}(Q_{s,L}, \mathcal{H}_{\mathrm{wav},\alpha,s,p}) \leq C N^{-(\alpha+1/2-1/p')} \log(N)^{\frac{s-1}{p'}}.$$

In particular,

$$e^{\mathrm{wor}}(Q_{s,L}, \mathcal{H}_{\mathrm{wav},\alpha,s,2}) \leq C N^{-\alpha} \log(N)^{\frac{s-1}{2}}.$$

Corollary. $\alpha \in (1/2, 1)$. Then

$$D^{*}_{\alpha,s,2}(X) = e^{\mathrm{wor}}(Q_{s,L}, \mathcal{H}_{\alpha,s,2}) \leq C N^{-\alpha} \log(N)^{\frac{s-1}{2}}.$$

Previously Known Results

X $(t, L+t, s)$-net in base b, $N := |X| = b^{-L-t}$.

$$Q_{s,L}(f) = \frac{1}{N} \sum_{\mathbf{x} \in X} f(x).$$

Theorem[Heinrich, Hickernell, Yue'04]. $\alpha > 1/2$.

$$\mathrm{rms}_{\mathrm{scr}}\, e^{\mathrm{wor}}(Q^{\mathrm{scr}}_{s,L}, \mathcal{H}_{\mathrm{wav},\alpha,s,2}) \leq O\big(N^{-\alpha} \log(N)^{\frac{s-1}{2}}\big).$$

Lower bound for arbitrary quadratures Q with N integration points:

$$e^{\mathrm{wor}}(Q, \mathcal{H}_{\mathrm{wav},\alpha,s,2}) \geq \Omega\big(N^{-\alpha} \log(N)^{\frac{s-1}{2}}\big).$$

Theorem[Entacher'97]. $\alpha > 1/2$.

$$e^{\mathrm{wor}}(Q_{s,L}, \mathcal{H}_{\mathrm{wav},\alpha,s,\infty}) \leq O\big(N^{-(\alpha-1/2)} \log(N)^{s-1}\big).$$

Thank you for your attention!

Haar Wavelet Spaces ($s = 1$)

$b \geq 2$. Let $\varphi := 1_{[0,1)}$,

$$\varphi_k^j(x) := b^{j/2}\varphi(b^j x - k), \quad j \in \mathbb{N}_0,\ k = 0, 1, \ldots, b^j - 1.$$

$V^j := \operatorname{span}\{\varphi_k^j \,|\, k \in \Delta_j\}$. Then $V^0 \subset V^1 \subset \cdots \subset L_2([0,1])$.

$W^0 := V^0$ and $W^j \oplus V^{j-1} = V^j$ for $j \in \mathbb{N}$. Put $\psi_{0,0}^0 := \varphi$.

Let $\{\psi_0, \psi_1, \ldots, \psi_{b-2}\}$ be orthonormal base of W^1. Then

$$\psi_{i,k}^j(x) := b^{(j-1)/2}\psi_i(b^{j-1}x - k) \quad j \in \mathbb{N},\ k \in \Delta_{j-1},\ i \in \nabla_j,$$

othonormal basis of W^j.

Embedding the Besov into the Wavelet Space

Want to show

$$\|f\|^2_{\mathrm{wav},\alpha,2} \leq C \left(\|f\|^2_{L_2} + \sum_{j=0}^{\infty} b^{2j\alpha} \omega_2(f, b^{-j})^2 \right).$$

Proof idea:

- If P_j orthogonal projection onto V^j, then $P_j - P_{j-1}$ orthogonal projection onto W^j. Hence

$$\|f\|^2_{\mathrm{wav},\alpha,2} = \sum_{j=0}^{\infty} b^{2\alpha j} \|(P_j - P_{j-1})f\|^2_{L_2}.$$

- $\|(P_j - P_{j-1})f\|^2_{L_2} \leq 2(\|P_j f - f\|^2_{L_2} + \|P_{j-1} f - f\|^2_{L_2})$.
- Jackson-type estimate:

$$\inf_{v \in V^j} \|f - v\|_{L_2} \leq 2^{1/2}\, \omega_2(f, b^{-j}) \quad \text{for } f \in L_2([0,1]),\ j \in \mathbb{N}_0.$$

Net Quadratures on Wavelet Spaces

Proof idea: Approximation and Wavelet space of level L

$$V^{s,L} := \sum_{|\mathbf{j}|=L} \otimes_{\ell=1}^{s} V^{j_\ell} \quad \text{and put} \quad W^{s,L} := \oplus_{|\mathbf{j}|=L} \otimes_{\ell=1}^{s} W^{j_\ell}.$$

Then $V^{s,L} = V^{s,L-1} \oplus W^{s,L}$ for $L \in \mathbb{N}$.

$(t, L+t, s)$-net X is exact on $V^{s,L}$. Thus

$$e^{\text{wor}}(Q_{s,L}, \mathcal{H}_{\text{wav},s,2}) \leq \sum_{|\mathbf{j}|_1=L+1}^{\infty} \sum_{\mathbf{k}\in\Delta_{\mathbf{j}-\mathbf{1}}} \sum_{\mathbf{i}\in\nabla_{\mathbf{j}}} b^{-2\alpha|\mathbf{j}|_1} (Q_{s,L}(\Psi_{\mathbf{i},\mathbf{k}}^{\mathbf{j}}))^2.$$