A Multilevel Monte Carlo Algorithm for Lévy Driven SDEs

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joint work with Steffen Dereich
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Outline

Setting

SDE

$$\begin{aligned} dY_t &= a(Y_{t-})\, dX_t, \qquad t \in [0,1], \\ Y_0 &= y_0, \end{aligned}$$

with

- an $\mathbb{R}^{d_X}$-valued square integrable Lévy process $X = (X_t)_{t\in[0,1]}$,
- a deterministic initial value $y_0 \in \mathbb{R}^{d_Y}$,
- a Lipschitz continuous function $a : \mathbb{R}^{d_Y} \to \mathbb{R}^{d_Y \times d_X}$.

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For simplicity: **Scalar autonomous SDE**

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Computational problem: Compute

$$S(f) = \mathbb{E}[f(Y)]$$

for functionals $f : D[0,1] \to \mathbb{R}$.

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Example: payoff f of a path dependent option.

Infinite dimensional quadrature

- Creutzig, Dereich, Müller-Gronbach, Ritter (2009)
- Kuo, Sloan, Wasilkowski, Woźniakowski (2010)
- Hickernell, Müller-Gronbach, Niu, Ritter (2010)
- Plaskota, Wasilkowski (2010)
- ...

See plenary talk by Wasilkowski.

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Basic idea:

- $Y^{(1)}, Y^{(2)}, \dots$ strong approximations for the solution Y with increasing accuracy and increasing numerical cost.
- Telescoping sum for the approximation of $\mathbb{E}[f(Y)]$:

$$\mathbb{E}[f(Y^{(m)})] = \mathbb{E}[\underbrace{f(Y^{(1)})}_{=D_1}] + \sum_{k=2}^{m} \mathbb{E}[\underbrace{f(Y^{(k)}) - f(Y^{(k-1)})}_{=D_k}]$$

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Note: $(Y^{(k)}, Y^{(k-1)})$ are coupled via X so that the variance of D_k decreases.

Lévy processes

$X = (X_t)_{t \geq 0}$ Lévy process, i.e.

- $X_0 = 0$,
- X *càdlàg* (right-continuous with left limits),
- *independent increments*, i.e. for $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \ldots < t_n < \infty$, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ independent,
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Examples:

- Brownian motion: $X_t - X_s \sim \mathcal{N}(0, t - s)$.
- Poisson process with parameter $\lambda > 0$: $X_t - X_s \sim Poi(\lambda(t - s))$.

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In the sequel: X square integrable.

Lévy-Ito decomposition of X

Idea: Cutoff of the small jumps of X. Denote $\Delta X_t = X_t - \lim_{s \nearrow t} X_s$.

For $A \in \mathfrak{B}(\mathbb{R})$ and $t \geq 0$,

$$N(t, A) = \#\{s \in [0, t] : \Delta X_s \in A\},$$
$$\nu(A) = \mathbb{E}[N(1, A)].$$

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Define for fixed $h > 0$

$$L_t^{(h)} = \underbrace{\int_{(-h,h)^c} x \, N(t, dx)}_{\stackrel{d}{=} \sum_{j=1}^{N_t} \xi_j} - t \underbrace{\int_{(-h,h)^c} x \, \nu(dx)}_{\in \mathbb{R}}.$$

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- $(N_t)_{t \geq 0}$ and $(\xi_j)_{j \in \mathbb{N}}$ independent.

Lévy-Ito decomposition of X

Facts:

- $(L_t^{(h)})_{t\geq 0}$ is an L_2-martingale.
- $(L_t)_{t\geq 0} = L_2\text{-}\lim_{h\to 0} L^{(h)}$ exists.
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Remark: $\mathbb{P}_X$ uniquely determined by drift b, Lévy measure ν and Gauss coefficient σ.

Approximation of X

Basic idea: Define parameters $h, \varepsilon > 0$.

- Approximate X by $(bt + \sigma W_t + L_t^{(h)})_{t \geq 0}$.
- Time discretization, $T_0^{(h,\varepsilon)} = 0$ and

$$T_j^{(h,\varepsilon)} = \inf \left\{ t > T_{j-1}^{(h,\varepsilon)} : |\Delta L_t^{(h)}| \neq 0 \right\} \wedge (T_{j-1}^{(h,\varepsilon)} + \varepsilon).$$

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Remark: Simulation requires samples according to $\mu^{(h)}$.

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Problem: Simulation of the coupled processes $(X^{(h,\varepsilon)}, X^{(h',\varepsilon')})$ for $h' > h > 0$ and $\varepsilon' > \varepsilon > 0$.

The coupled simulation I

Simulation of the jump part of $L^{(h)}$: Given $h' > h > 0$ and $\varepsilon' > \varepsilon > 0$.

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Hence, for $(L^{(h)}, L^{(h')})$,

$$L_t^{(h)} = \sum_{s \in [0,t]} \Delta L_s^{(h)} - t \int_{(-h,h)^c} x \, \nu(dx),$$

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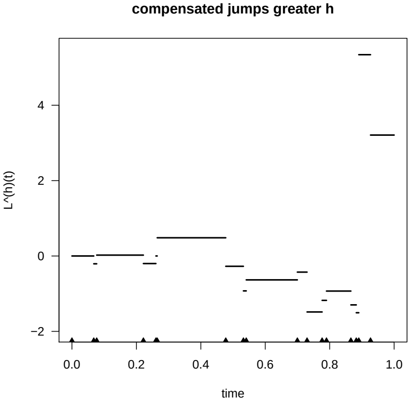
Compensated compound Poisson processes $(L^{(h)}, L^{(h')})$



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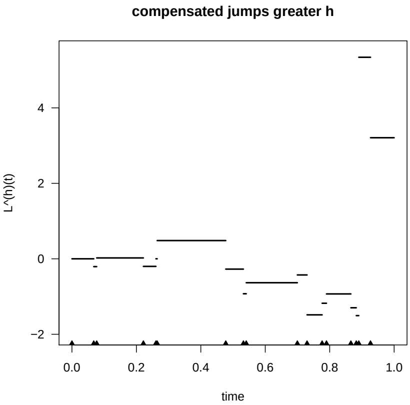

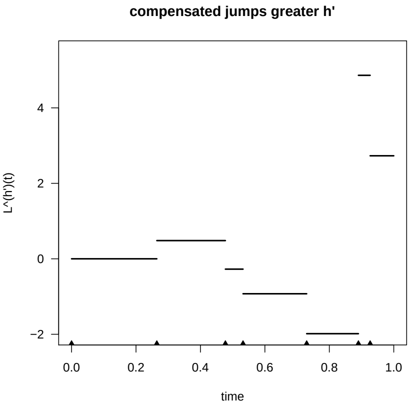

The coupled simulation II

Refine the time grids

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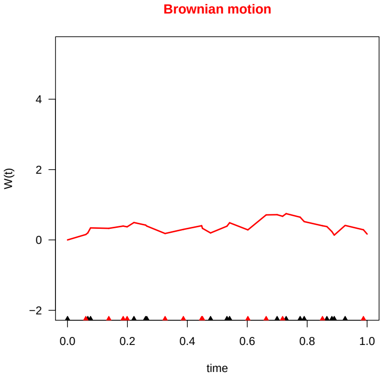
Simulate W at $\{T_j^{(h,\varepsilon)} : j \in \mathbb{N}\} \cup \{T_j^{(h',\varepsilon')} : j \in \mathbb{N}\}$.



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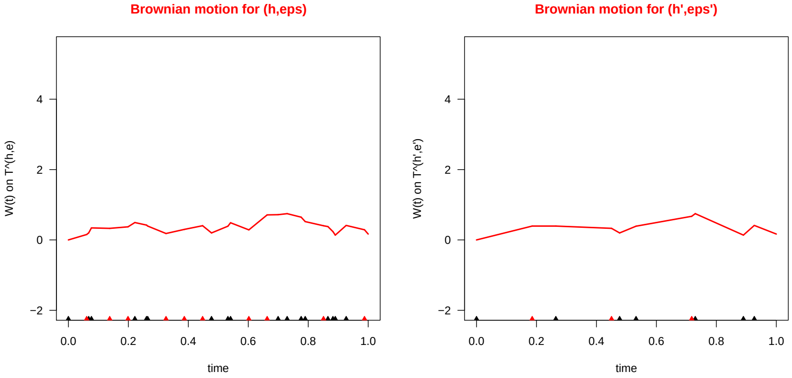
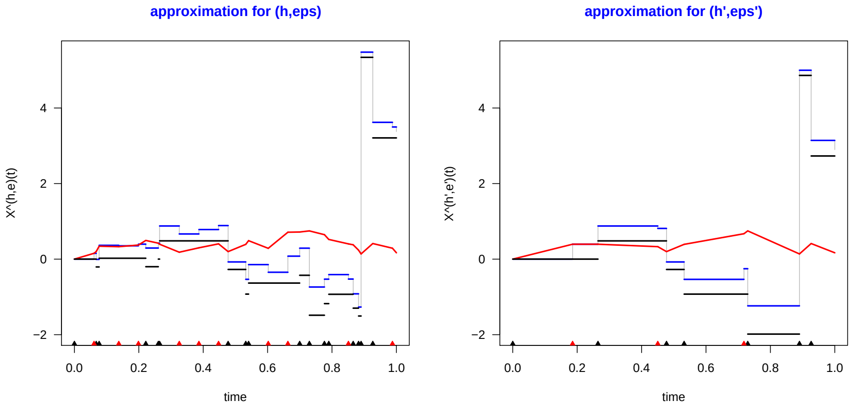
Coarsening W to $(T_j^{(h,\varepsilon)})$ and $(T_j^{(h',\varepsilon')})$, respectively.



The piecewise constant approximations



The Multilevel Euler Algorithm

Recall: $$\mathbb{E}[f(Y^{(m)})] = \mathbb{E}[f(Y^{(1)})] + \sum_{k=2}^{m} \mathbb{E}[f(Y^{(k)}) - f(Y^{(k-1)})]$$

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Free parameters of the algorithm $\hat{S}(f)$:

- $h_1 > h_2 > \ldots$ (approximation of L)
- $\varepsilon_1 > \varepsilon_2 > \ldots$ (approximation of W)
- m (number of levels)
- $n_1, \ldots, n_m$ (number of replications per level)

Error and Cost

Error: $$e^2(\hat{S}) = \sup_{f \in \mathrm{Lip}(1)} \mathbb{E}[|S(f) - \hat{S}(f)|^2]$$

Here: $\mathrm{Lip}(1) := \{f : D[0,1] \to \mathbb{R} \mid f \ 1 - \text{Lipschitz w.r.t. supremum norm}\}$

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Cost: $$\mathrm{cost}(\hat{S}) \asymp \sum_{k=1}^{m} n_k \mathbb{E}[\#\mathrm{breakpoints}(Y^{(k)}) + 1]$$

Result (Dereich, H.)

Theorem

For driving Lévy process with **Blumenthal-Getoor-index**

$$\beta := \inf\left\{p > 0 : \int_{(-1,1)} |x|^p \, \nu(dx) < \infty\right\} \in [0,2]$$

there exist algorithms $(\hat{S}_n)_{n\in\mathbb{N}}$ *with* $\mathrm{cost}(\hat{S}_n) \precsim n$ *such that*

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- Using a Gaussian approximation for the neglected small jumps improves the result for $\beta \geq 1$ (see Dereich, 2010).

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- $Y_t = B_{T_t}, t \geq 0$.
- Time change a Brownian motion B with an independent Gamma subordinator T.
- Gamma process T has marginal distribution density

$$f_{T_t}(x) = \mathbb{1}_{\mathbb{R}_+}(x) \frac{\theta^{rt}}{\Gamma(rt)} x^{rt-1} \exp(-\theta x), \quad t \geq 0.$$

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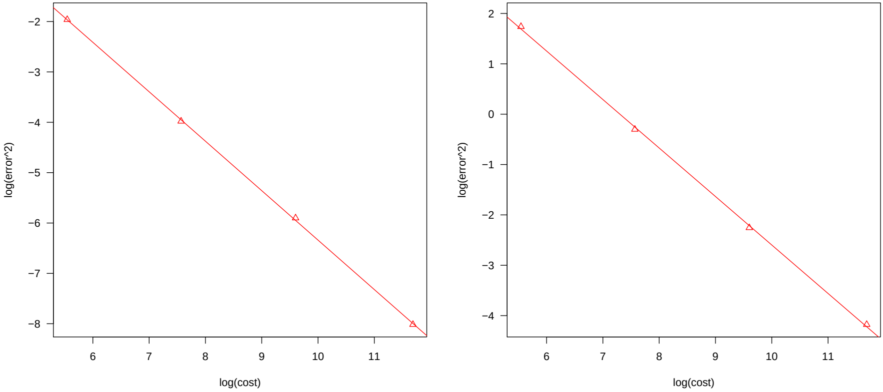
- Infinite activity process with Blumenthal-Getoor-index $\beta = 0$.
- Theorem provides $e(\hat{S}_n) \precsim n^{-1/2}$.

Example: Variance Gamma

Variance Gamma process Y with parameters $r, \theta > 0$

- Simulations with $\theta = r = 2$, $a(x) = 2x$ and $y_0 = 1$.
- MLMC for $f(X) = min_{t\in[0,1]}(X_t)$ (left) and $f(X) = max_{t\in[0,1]}(X_t)$ (right). Regression slopes: 0.49



Example: α-stable processes

α-**stable process** X with index of stability $\alpha \in (0, 2]$

- self-similarity $\left(\frac{X_{at}}{a^{1/\alpha}}\right)_{t \geq 0} \stackrel{d}{=} (X_t)_{t \geq 0}$.

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Example: α-stable processes

Truncated symmetric α-stable processes

- Simulations with $a(x) = 2x, y_0 = 1$ and $c = 0.1$
- MC and MLMC for $\alpha = 1.1, u = 4$ and $f(X) = \max_{t\in[0,1]}(X_t)$ (left) and $\alpha = 0.5, u = 4$ and $f(X) = \min_{t\in[0,1]}(X_t)$ (right).