Randomized Approximation of Functions with Applications to Elliptic PDEs

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1. Notation

$1 \le p \le \infty$, $d \in \mathbb{N} = \{1, 2, \dots\}$, $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $Q \subset \mathbb{R}^d$ a bounded Lipschitz domain

Sobolev space

$$W_p^r(Q) = \{f \in L_p(Q) : \ D^\alpha f \in L_p(Q), \ |\alpha| \le r\}$$

$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| := \sum_{j=1}^d \alpha_j \le r$,
$D^\alpha f$ generalized partial derivative

norm

$$\|f\|_{W_p^r(Q)} = \left(\sum_{|\alpha| \le r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \le r} \|D^\alpha f\|_{L_\infty(Q)}.$$

Note:

$$r = 0 \quad W_p^0(Q) = L_p(Q)$$

Approximation problem

$1 \le p, q \le \infty,\ r, s \in \mathbb{N}_0,$

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right)$$

Approximate

$$J : W_p^r(Q) \to W_q^s(Q)$$

deterministic algorithms $\mathcal{A}_n^{\mathrm{det}}$

$$A : W_p^r(Q) \to W_q^s(Q),$$

$$A(f) = \sum_{i=1}^{n} f(x_i)\psi_i \quad x_i \in Q,\ \psi_i \in W_q^s(Q)$$

error:

$$
\begin{aligned}
& e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \\
& \quad = \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - A(f)\|_{W_q^s(Q)}
\end{aligned}
$$

deterministic n-th minimal error:
(linear sampling numbers)

$$
\begin{aligned}
& e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \\
& \quad = \inf_{A \in \mathcal{A}_n^{\mathrm{det}}} e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))
\end{aligned}
$$

randomized algorithms $\mathcal{A}_n^{\text{ran}}$:

$(\Omega, \Sigma, \mathbb{P})$ probability space,

$$(A_\omega)_{\omega \in \Omega}, \qquad A_\omega : W_p^r(Q) \to W_q^s(Q)$$

$$A_\omega(f) = \sum_{i=1}^{n} f(x_{i,\omega}) \psi_{i,\omega}$$

$x_{i,\omega} \in Q$, $\psi_{i,\omega} \in W_q^s(Q)$ $(\omega \in \Omega)$,

error:

$$e(J, (A_\omega), \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) = \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \mathbb{E} \, \|Jf - A_\omega(f)\|_{W_q^s(Q)}$$

randomized n-th minimal error:

(randomized linear sampling numbers)

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) = \inf_{(A)_\omega \in \mathcal{A}_n^{\text{ran}}} e(J, (A)_\omega, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q)) \le e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^s(Q))$$

e_n^{det} : $\quad \Omega = \{\omega_0\}$

embedding condition

$W_p^r(Q)$ is embedded into $C(Q)$ iff

$$\left.\begin{array}{lll} & p = 1 & \text{and} \quad r/d \ge 1 \\ \text{or} & & \\ & 1 < p \le \infty & \text{and} \quad r/d > 1/p \end{array}\right\} \quad (1)$$

2. **Embedding** $J : W_p^r(Q) \to W_q^s(Q)$ **with** $s \geq 0$

Theorem 1. *(many authors)*
Let $r, s \in \mathbb{N}_0$, $1 \leq p, q \leq \infty$, *with*

$$\frac{r-s}{d} > \max\left(\frac{1}{p} - \frac{1}{q}, 0\right),$$

let Q *be a bounded Lipschitz domain. Then in the deterministic setting, if the embedding condition (1) holds,*

$$e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_p^s(Q)) \asymp n^{-\frac{r-s}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+},$$

if the embedding condition does not hold, then

$$e_n^{\text{det}}(J, \mathcal{B}_{W_p^r(Q)} \cap C(Q), W_p^s(Q)) \asymp 1.$$

In the randomized setting

$$e_n^{\text{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_p^s(Q)) \asymp n^{-\frac{r-s}{d} + \left(\frac{1}{p} - \frac{1}{q}\right)_+}.$$

independently of the embedding condition.

deterministic, Q bounded Lipschitz domain, $s > 0$: H. 2008

solved Problem 18 of Novak, Woźniakowski (Tractability of Multivariate Problems, Volume 1)

3. Embedding into spaces with negative smoothness

$J : W_p^r(Q) \to W_q^{-s}(Q)$. We define

$$W_q^{-s}(Q) := \widetilde{W}_{q^*}^s(Q)^* \qquad \left(\frac{1}{q} + \frac{1}{q^*} = 1\right)$$

where $\widetilde{W}_{q*}(Q)$ is the closure of C^∞ functions with support in Q, in the norm of $W_{q*}(Q)$, and $\widetilde{W}_{q^*}^s(Q)^*$ is its dual space.

Motivation 1: weak elliptic problem

$m \in \mathbb{N}$, bilinear form a on $W_2^m(Q)$,

$$a(u,v) = \sum_{|\alpha|,|\beta| \le m} \int_Q a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) dx,$$

$a_{\alpha\beta} \in C(Q)$

assume a is $\widetilde{W}_2^m(Q)$-elliptic

$$\begin{aligned} |a(u,v)| &\le c_1 \|u\|_{W_2^m(Q)} \|v\|_{W_2^m(Q)} \\ a(u,u) &\ge c_2 \|u\|^2_{W_2^m(Q)} \end{aligned}$$

$(u, v \in \widetilde{W}_2^m(Q))$.

associated differential operator:

$$\mathscr{L} u = \sum_{|\alpha|,|\beta| \leq m} D^{\beta}(a_{\alpha\beta}\, D^{\alpha} u)$$

associated differential equation (under additional smoothness): given f, find u with

$$\begin{aligned} \mathscr{L} u &= f \quad \text{on} \quad Q \\ D_n^{\gamma} u &= 0 \quad \text{on} \quad \partial Q \quad (0 \leq \gamma \leq m-1) \end{aligned}$$

weak elliptic problem associated with a:

Given $f \in W_2^{-m}(Q)$, find $u \in \widetilde{W}_2^m(Q)$ such that for all $v \in \widetilde{W}_2^m(Q)$

$$a(u, v) = f(v).$$

The problem has a unique solution $u = S_0 f \in \widetilde{W}_2^m(Q)$,

$$S_0 : W_2^{-m}(Q) \to \widetilde{W}_2^m(Q)$$

is an isomorphism

$r \in \mathbb{N}_0$, solve the weak problem for $f \in W_2^r(Q)$: solution operator

$$S^{\mathrm{ell}} = S_0 J : W_2^r(Q) \xrightarrow{J} W_2^{-m}(Q) \xrightarrow{S_0} \widetilde{W}_2^m(Q)$$

Corollary 1.

$$\begin{aligned} & e_n^{\mathrm{ran}}(S^{\mathrm{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \\ \asymp \; & e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_2^r(Q)}, W_2^{-m}(Q)), \end{aligned}$$

and analogously for e_n^{det}.

Motivation 2:
Simultaneous quadrature over classes of weights

deterministic case:

$$e(J, A, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q))$$

$$= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - A(f)\|_{W_q^{-s}(Q)}$$

$$= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \|Jf - \sum_{i=1}^{n} f(x_i)\psi_i\|_{W_q^{-s}(Q)}$$

$$= \sup_{f \in \mathcal{B}_{W_p^r(Q)}} \sup_{g \in \mathcal{B}_{\widetilde{W}_{q^*}^s(Q)}} \left| \int_Q f(x)g(x)dx - \sum_{i=1}^{n} f(x_i)(\psi_i, g) \right|$$

$$\int_Q f(x)g(x)dx \approx \sum_{i=1}^{n} (\psi_i, g) f(x_i)$$

quadrature, the quadrature weights depending on the integration weight g only through n linear functionals (information functionals) randomized case:

$$e(J,(A_\omega),\mathcal{B}_{W_p^r(Q)},W_q^s(Q))$$

$$= \sup_{f\in\mathcal{B}_{W_p^r(Q)}} \mathbb{E}\,\|Jf - A_\omega(f)\|_{W_q^{-s}(Q)}$$

$$= \sup_{f\in\mathcal{B}_{W_p^r(Q)}} \mathbb{E}\,\|Jf - \sum_{i=1}^{n} f(x_{i,\omega})\psi_{i,\omega}\|_{W_q^{-s}(Q)}$$

$$= \sup_{f\in\mathcal{B}_{W_p^r(Q)}}$$

$$\mathbb{E} \sup_{g\in\mathcal{B}_{\widetilde{W}^s_{q^*}(Q)}} \left|\int_Q f(x)g(x)dx - \sum_{i=1}^{n} f(x_{i,\omega})(\psi_{i,\omega},g)\right|$$

error criterion: uniformly over the class of weights

Deterministic case:

Theorem 2. *Let* $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $1 \le p, q \le \infty$,

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

1. (Novak, Triebel, 2006, Vybiral, 2007) Assume that $W_p^r(Q)$ *is embedded into* $C(Q)$*. Then*

$$e_n^{\mathrm{det}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp n^{-\gamma_1}$$

where

$$\gamma_1 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d}\right).$$

2. (H., 2008) If $W_p^r(Q)$ *is not embedded into* $C(Q)$*, then*

$$e_n^{\mathrm{det}}(J : \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp 1.$$

Randomized case:

Theorem 3. *(H., 2008) Let* $r \in \mathbb{N}_0$, $s \in \mathbb{N}$, $1 \le p, q \le \infty$ *and*

$$\frac{r+s}{d} > \frac{1}{p} - \frac{1}{q}.$$

Then

$$e_n^{\mathrm{ran}}(J, \mathcal{B}_{W_p^r(Q)}, W_q^{-s}(Q)) \asymp_{log} \; n^{-\gamma_2}$$

where

$$\gamma_2 = \min\left(\frac{r+s}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)_+, \frac{r}{d} + 1 - \frac{1}{\bar{p}}\right)$$

and $\bar{p} = \min(p, 2)$.

solves Problem 25 of Novak, Woźniakowski (Tractability of Multivariate Problems, Volume 1)

For elliptic PDEs this yields the deterministic and randomized n-th minimal error:

Corollary 2. *Let* $r \in \mathbb{N}_0$, $m \in \mathbb{N}$. *Then*

$$e_n^{\mathrm{det}}(S^{\mathrm{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \asymp n^{-\frac{r}{d}}$$

$$e_n^{\mathrm{ran}}(S^{\mathrm{ell}}, \mathcal{B}_{W_2^r(Q)}, \widetilde{W}_2^m(Q)) \asymp n^{-\frac{r}{d}-\min\left(\frac{m}{d},\frac{1}{2}\right)}.$$

Thus, randomization gives a speedup of $\min\left(\frac{m}{d},\frac{1}{2}\right)$ in the exponent.

4. The algorithm

We assume for simplicity $1 < p = q < \infty$, $Q = [0,1]^d$,

$$J = J_2 J_1 : W_p^r(Q) \xrightarrow{J_1} L_p(Q) \xrightarrow{J_2} W_p^{-s}(Q)$$

duality

$$\begin{aligned} J_2 &: L_p(Q) \to W_p^{-s}(Q) = \widetilde{W}_{p^*}^s(Q)^* \\ J_0 &: \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q) \qquad J_2 = J_0^* \end{aligned}$$

$$\|J_0 - P_k : \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q)\| \le c2^{-sk}$$

$$P_k h = \sum_{i=1}^{n_k} (h, \tilde{h}_{ki}) \tilde{g}_{ki},$$

$n_k \asymp 2^{dk}$, $\tilde{h}_{ki} \in \widetilde{W}_{p^*}^s(Q)^*$,

$\tilde{g}_{ki} \in L_{p^*}(Q)$ almost disjoint supports

$$\|J_0^* - P_k^* : L_p(Q) \to W_p^{-s}(Q)\| \le c2^{-sk}$$

$$P_k^* f = \sum_{i=1}^{n_k} (f, \tilde{g}_{ki}) \tilde{h}_{ki},$$

idea: use simultaneous Monte Carlo integration for the approximation of the weighted integrals

$$(f, \tilde{g}_{ki}) = \int_Q f(x) \tilde{g}_{ki}(x) dx$$

Does not give the optimal rate!

Multilevel splitting:

$$P_k = \sum_{l=0}^{k} (P_l - P_{l-1}), \qquad P_{-1} = 0$$

$$\|P_l - P_{l-1} : \widetilde{W}_{p^*}^s(Q) \to L_{p^*}(Q)\| \le c2^{-sk}$$

$$(P_l - P_{l-1})h = \sum_{i=1}^{n_l} (h, h_{li}) g_{li}, \quad h_{li} \in \widetilde{W}^s_{p^*}(Q)^*,$$

$\Rightarrow$ approximate

$$(P_l^* - P_{l-1}^*)f = \sum_{i=1}^{n_l} (f, g_{li}) h_{li},$$

Assumptions: For $u = p^*, \infty$

$$c_1 n_l^{-1/u} \|(\lambda_i)\|_{\ell_u^{n_l}} \le \left\| \sum_{i=1}^{n_l} \lambda_i g_{li} \right\|_{L_u(Q)} \le c_2 n_l^{-1/u} \|(\lambda_i)\|_{\ell_u^{n_l}}$$

$$\operatorname{supp} g_{li} \subseteq Q_{li}, \quad |Q_{li}| \le c_3 n_l^{-1},$$

$$\max_{x \in Q} |\{i : x \in Q_{li}\}| \le c_4 \quad (Q_{li} \text{ almost disjoint}).$$

ξ_{li} uniformly distributed on Q_{li}

$$\gamma_{li}(f) := |Q_{li}| g_{li}(\xi_{li}) f(\xi_{li})$$

$$\mathbb{E}\,\gamma_{li}(f) = \int_{Q_{li}} g_{li}(x) f(x) dx = (g_{li}, f)$$

$$\begin{aligned} (P_l^* - P_{l-1}^*) f &= \sum_{i=1}^{n_l} (f, g_{li}) h_{li} \\ \approx \;\; A_{l,N_l}(f) &= \sum_{i=1}^{n_l} \left(\frac{1}{N_l} \sum_{j=1}^{N_l} \gamma_{lij}(f) \right) h_{li} \end{aligned}$$

$\gamma_{lij}(f)$ independent copies of $\gamma_{li}(f)$.

Proposition 1. *Let* $\bar{p} = \min(2, p)$. *Then*

$$\begin{aligned} &\left(\mathbb{E}\, \| f - A_{l,N_l}(f) \|_{W_p^{-s}}^{\bar{p}} \right)^{1/\bar{p}} \\ &\quad \leq \;\; c 2^{-sl} N_l^{-1+1/\bar{p}} \| f \|_{L_p(Q)}. \end{aligned}$$

Final approximations:

$$
\begin{aligned}
J_2 f &\approx A_k f \\
&= \sum_{l=0}^{k} \sum_{i=1}^{n_l} \left(\frac{1}{N_l} \sum_{j=1}^{N_l} |Q_{li}| g_{li}(\xi_{lij}) f(\xi_{lij}) \right) h_{li}
\end{aligned}
$$

$$\text{Total error} \le c \sum_{l=0}^{k} 2^{-sl} N_l^{-1+1/\bar{p}}$$

$$\text{cost} \le c \sum_{l=0}^{k} 2^{dl} N_l$$

$$Jf \approx U_k f + A_k(f - U_k f),$$

where

$$(\,\mathbb{E}\,)\, \|f - U_k f\|_{L_p(Q)} \le c\, 2^{-rk} \|f\|_{W_p^r(Q)}$$

5. Sobolev spaces of functions with dominating mixed derivatives

$Q = [0,1]^d$, $1 \le p \le \infty$,

$$\bar{1} = (\underbrace{1, 1, \dots, 1}_{d \text{ times}})$$

$$W_p^{\bar{1}}(Q) = \{f \in L_p(Q) : \ D^{\bar{1}} f \in L_p(Q)\}$$

$$\widehat{W}_p^{\bar{1}}(Q) = \{f \in W_p^{\bar{1}}(Q) : f(x_1, \dots, x_d) = 0 \text{ if } \exists i : x_i = 0\}$$

norm

$$\|f\|_{\widehat{W}_p^{\bar{1}}(Q)} = \|D^{\bar{1}} f\|_{L_p(Q)}$$

Negative smoothness: We define

$$\check{W}_p^{-\bar{1}}(Q) = \widehat{W}_{p^*}^{\bar{1}}(Q)^*$$

Problem: For $1 \le p, q \le \infty$ approximate

$$J : L_p(Q) \to \check{W}_q^{-\bar{1}}(Q)$$

Motivation 1: Indefinite integration

$$S : L_p\,(Q) \to L_q(Q) \qquad Q = [0,1]^d$$

given by

$$\begin{aligned}(Sf)(x) &= \int\limits_{[0,x]} f(t)dt \\ &= \int_0^{x_1} \dots \int_0^{x_d} f(t_1, \dots, t_d)dt,\end{aligned}$$

where $x = (x_1, \dots, x_d) \in Q$,
$[0, x] = [0, x_1] \times ... \times [0, x_d]$

equivalence to approximation:

$$S : L_{q^*}(Q) \to \widehat{W}^{\bar{1}}_{q^*}(Q)$$

is an isometric isomorphism, hence

$$\begin{aligned} S^* &: \check{W}_q^{-\bar{1}}(Q) \to L_q(Q) \\ (S^*)^{-1} &: L_q(Q) \to \check{W}_q^{-\bar{1}}(Q)\end{aligned}$$

are isometric isomorphisms, as well.

$$J : L_p(Q) \xrightarrow{S^*} L_q(Q) \xrightarrow{(S^*)^{-1}} \check{W}_q^{-\bar{1}}(Q)$$

Corollary 3.

$$\begin{aligned} & e_n^{\text{ran}}(J, \mathcal{B}_{L_p(Q)}, \check{W}_q^{-\bar{1}}(Q)) \\ = \; & e_n^{\text{ran}}(S^*, \mathcal{B}_{L_p(Q)}, L_q(Q)) \\ = \; & e_n^{\text{ran}}(S, \mathcal{B}_{L_p(Q)}, L_q(Q)) \end{aligned}$$

and analogously for e_n^{det}.

(one gets S^* from S by change of variables)

Motivation 2: Simultaneous quadrature over classes of weights

Compute

$$\int_Q f(x)g(x)dx$$

for $f \in L_p(Q)$, $g \in \widehat{W}_p^{\bar{1}}(Q)$

From now on we consider indefinite integration

$$S : L_p(Q) \to L_q(Q)$$

Theorem 4. *(H., Milla, 2010)*
Let $1 \le p, q \le \infty$ *and* $\bar{p} = \min(p, 2)$*. Then*

$$e_n^{\mathrm{ran}}(S, \mathcal{B}_{L_p(Q)}, L_q(Q)) \asymp n^{-1+1/\bar{p}}.$$

Two algorithms:

- Smolyak-Monte Carlo algorithm

$$\begin{aligned} Sf &\approx P_L^{(d)} Sf = \sum_{l=0}^{L} (P_l - P_{l-1}) \otimes P_{L-l}^{(d-1)} Sf \\ &= \sum_{l_1+l_2+\cdots+l_d=L} (P_{l_1} - P_{l_1-1}) \otimes \ldots \\ &\qquad \ldots \otimes (P_{l_{d-1}} - P_{l_{d-1}-1}) \otimes P_{l_d} Sf \end{aligned}$$

  Then apply grid-stratified sampling to obtain an approximation to the needed values $(Sf)(x_i)$

  norm estimates use Banach space tensor products (related to p-nuclear operators)

- Simple sampling algorithm:

  $(\xi_i)_{i=1}^n$ independent, uniformly distributed on $Q = [0,1]^d$

$$
\begin{aligned}
(Sf)(x) &= \int \chi_{[0,x]} f(x) dx \\
&\approx (A_n f)(x) = \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,x]}(\xi_i) f(\xi_i) \\
& \qquad\qquad (x \in Q)
\end{aligned}
$$

  thus

$$
Sf \approx A_n f = \frac{1}{n} \sum_{i=1}^{n} f(\xi_i) \chi_{[\xi_i, \bar{1}]}
$$

6. Estimates with polynomial dependence on d (tractability)

From now on we consider $q = \infty$. In this case the problem is normalized:

$$\|S : L_p(Q) \to L_\infty(Q)\| = 1.$$

Theorem 5. *(H., Milla, 2010) Let $1 \le p \le \infty$, and $\bar{p} = \min(p, 2)$. Then there is a constant $c > 0$ such that for all $d, n \in \mathbb{N}$, $f \in L_p(Q)$, the simple sampling algorithm A_n satisfies*

$$\mathbb{E}\, \|Sf - A_n f\|_{L_\infty(Q)} \le c d^{1-1/\bar{p}} n^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.$$

Idea of proof: $m \in \mathbb{N}$, Γ_m equidistant grid on $Q = [0,1]^d$ with mesh-size $1/m$.

Σ_m the σ-algebra generated by $\{[0,x] : x \in \Gamma_m\}$

$M(Q, \Sigma_m)$ Banach space of signed measures on Σ_m, with the total variation norm.

Define

$$\eta_i(B) = \int_B f(t)dt - \chi_B(\xi_i)f(\xi_i) \qquad (B \in \Sigma_m)$$

– independent $M(Q, \Sigma_m)$-valued random variables of zero mean.

The operator

$$\begin{aligned} J_m &: M(Q, \Sigma_m) \to \ell_\infty(\Gamma_m), \\ J_m\mu &= (\mu([0,x]))_{x \in \Gamma_m} \end{aligned}$$

has type $\bar{p}$ constant $\tau_{\bar{p}}(J_m)$

$$\tau_{\bar{p}}(J_m) \leq cd^{1-1/\bar{p}}$$

(Pisier, 1984)

(this uses the fact that the Vapnik-Červonenkis dimension of the family of sets $\{[0,x] : x \in [0,1]^d\}$ is d)

$$
\begin{aligned}
&\mathbb{E} \sup_{x\in\Gamma_m} \left| \int_{[0,x]} f(t)dt - \frac{1}{n}\sum_{i=1}^n \chi_{[0,x]}(\xi_i) f(\xi_i) \right| \\
\leq\;& n^{-1} \left(\mathbb{E} \Big\| \sum_{i=1}^n J_m \eta_i \Big\|_{\ell_\infty(\Gamma_m)}^{\bar{p}} \right)^{1/\bar{p}} \\
\leq\;& c d^{1-1/\bar{p}} n^{-1} \left(\sum_{i=1}^n \mathbb{E} \, \|\eta_i\|_{M(Q,\Sigma_m)}^{\bar{p}} \right)^{1/\bar{p}} \\
\leq\;& c d^{1-1/\bar{p}} n^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.
\end{aligned}
$$

$m \to \infty$, density argument

7. A d-dependent lower bound

Using an argument by A. Hinrichs, we obtain a d-dependent (not sharp in n) lower bound:

Theorem 6. *(H., Milla, 2010) Let* $1 \leq p \leq \infty$*. There is a constant* $c > 0$ *such that for all* $d, n \in \mathbb{N}$

$$e_n^{\mathrm{ran}}(S, \mathcal{B}_{L_p([0,1]^d)}, L_\infty([0,1]^d)) \;\geq\; \min(d/n, 1).$$

Corollary 4. *Let* $1 \leq p \leq \infty$ *and* $\bar{p} = \min(2, p)$*. There are constants* $c_1, c_2 > 0$ *such that for all* $d, n \in \mathbb{N}$

$$\begin{aligned} c_1 \min(d/n, 1) \;&\leq\; e_n^{\mathrm{ran}}(S, \mathcal{B}_{L_p([0,1]^d)}, L_\infty([0,1]^d)) \\ &\leq\; c_2 d^{1-1/\bar{p}} n^{-(1-1/\bar{p})}. \end{aligned}$$

For $\varepsilon > 0$ the information complexity is defined as

$$n_\varepsilon^{\text{ran}}(S, \mathcal{B}_{L_p(Q)}, L_\infty(Q)) = \min\{n \in \mathbb{N}_0 : e_n^{\text{ran}}(S, \mathcal{B}_{L_p(Q)}, L_\infty(Q)) \le \varepsilon\}$$

Corollary 5. *Let* $1 < p \le \infty$ *and* $\bar{p} = \min(2, p)$*. There are constants* $c_1, c_2 > 0$*,* $\varepsilon_0 > 0$ *such that for all* $d \in \mathbb{N}, 0 < \varepsilon \le \varepsilon_0$

$$c_1 d/\varepsilon \;\le\; n_\varepsilon^{\text{ran}}(S, \mathcal{B}_{L_p(Q)}, L_\infty(Q)) \le c_2 d/\varepsilon^{\bar{p}/(\bar{p}-1)}.$$

8. Estimates of discrepancy with polynomial dependence on d (tractability of discrepancy)

From Theorem 5: Let $1 \le p \le \infty$. There is a constant $c > 0$ such that for all $d, n \in \mathbb{N}$, $f \in L_p(Q)$,

$$\begin{aligned}
&\mathbb{E}\, \|Sf - A_n f\|_{L_\infty(Q)} \\
&\quad = \mathbb{E} \sup_{x \in [0,1]^d} \left| \int_{[0,x]} f(t)dt - \frac{1}{n} \sum_{i=1}^{n} f(\xi_i) \chi_{[0,x]}(\xi_i) \right| \\
&\quad \le cd^{1-1/\bar{p}} n^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.
\end{aligned}$$

Corollary 6. *Let $1 \le p \le \infty$ and $\bar{p} = \min(2, p)$. Then there is a constant $c > 0$ such that for all $d, n \in \mathbb{N}$ and all $f \in L_p([0,1]^d)$ there exist $t_1, \dots, t_n \in [0,1]^d$ with*

$$\begin{aligned}
\sup_{x \in [0,1]^d} &\left| \int_{[0,x]} f(t)dt - \frac{1}{n} \sum_{i=1}^{n} f(t_i) \chi_{[0,x]}(t_i) \right| \\
&\le cd^{1-1/\bar{p}} n^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.
\end{aligned}$$

Taking $p = \infty$ and $f \equiv 1$, we recover the main result of

S. Heinrich, E. Novak, G. W. Wasilkowski, H. Woźniakowski, The inverse of the star-discrepancy depends linearly on the dimension, Acta Arithmetica 96 (2001), 279-302

Corollary 7. *There is a constant* $c > 0$ *such that for all* $d, n \in \mathbb{N}$ *there exist* $t_1, \ldots, t_n \in [0,1]^d$ *with*

$$\underbrace{\sup_{x \in [0,1]^d} \left| |[0,x]| - \frac{1}{n} \sum_{i=1}^{n} \chi_{[0,x]}(t_i) \right|}_{d_\infty^*(t_1,\ldots,t_n)} \leq c d^{1/2} n^{-1/2}.$$

Defining for any $1 \le p \le \infty$ and any $f \in L_p([0,1]^d)$ a generalization of the star-discrepancy as follows

$$d^*_{f,\infty}(t_1,\ldots,t_n)$$
$$= \sup_{x\in[0,1]^d} \left| \int_{[0,x]} f(t)dt - \frac{1}{n}\sum_{i=1}^{n} f(t_i)\chi_{[0,x]}(t_i) \right|,$$

Corollary 6 reads as follows

Corollary 8. *Let $1 \le p \le \infty$ and $\bar{p} = \min(2,p)$. Then there is a constant $c > 0$ such that for all $d, n \in \mathbb{N}$ and all $f \in L_p([0,1]^d)$ there exist $t_1,\ldots,t_n \in [0,1]^d$ with*

$$d^*_{f,\infty}(t_1,\ldots,t_n) \le cd^{1-1/\bar{p}} n^{-1+1/\bar{p}} \|f\|_{L_p(Q)}.$$

References

For the results of Sections 2–4 see

S. Heinrich, Randomized approximation of Sobolev embeddings II, J. Complexity 25 (2009), 455–472,

S. Heinrich, Randomized approximation of Sobolev embeddings III, J. Complexity 25 (2009), 473–507,

and the references therein. For the results of Sections 5-8 see

S. Heinrich, B. Milla, The randomized complexity of indefinite integration, J. Complexity (accepted)

and the references therein. Preprints of these papers can also be found on

http://www.uni-kl.de/AG-Heinrich/Publications.html