Uniform distribution of sequences in terms of $\mathbf{p}$-adic arithmetic

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Hybrid sequences

Recent developments

Niederreiter (2009, 2010) has established the first deterministic discrepancy bounds for various hybrid sequences.

Definition

Hybrid sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ in $[0, 1[^s$: mix at least two lower-dimensional sequences of different types s.t. certain coordinates of $\mathbf{x}_n$ stem from the first sequence, certain of the remaining coordinates stem from the second sequence, and so on.

History

Spanier (1995): the essential idea to combine MC and QMC sequences
Ökten (1996): probabilistic results on discrepancy
Ökten, Tuffin, Burago (2006): further probabilistic results
Gnewuch (2009): refinement of preceeding results

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The setting

The underlying space

$\mathbb{R}^s/\mathbb{Z}^s$ … s-dimensional torus, identified with $[0,1[^s$, $s \geq 1$.

The general goals

- prove uniform distribution of a given sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1[^s$
- measure uniform distribution of a given sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$
- construct low discrepancy sequences (LDS) ω

The central concepts

- discrepancy
- Weyl sums and Weyl criterion
- diaphony
- inequality Erdös-Turán-Koksma

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Discrepancy

Notation

$\omega = (\mathbf{x}_n)_{n \geq 0}$ ... sequence in $[0,1[^s$
$f : [0,1[^s \to \mathbb{R}(\mathbb{C})$ ... Riemann-integrable function

$$S_N(f,\omega) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

Definition

$\omega = (\mathbf{x}_n)_{n \geq 0}$ uniformly distributed in $[0,1[^s$ (u.d.), if

$$\lim_{N \to \infty} S_N(\mathbf{1}_J - \lambda(J), \omega) = 0 \qquad \forall J \text{ subinterval of } [0,1[^s$$

($\mathbf{1}_J$ ... indicator function of J, $\lambda(J)$ ... Lebesgue measure of J)

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$$D_N(\omega) = \sup_{J \text{ subinterval}} |S_N(\mathbf{1}_J - \lambda(J), \omega)|$$

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Weyl sums

Theorem (Weyl criterion)

$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1[^s$: $\quad e_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^{s} e^{2\pi i k_j x_j}$.

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$S_N(e_{\mathbf{k}}, \omega)$ . . . *Weyl sum (of order* $\mathbf{k}$*)*

Theorem (Inequality of Erdös-Turán-Koksma)

$$D_N(\omega) \;\leq\; C_s \left(\frac{1}{M} + \sum_{\mathbf{k} \in \Delta^*(M)} \frac{1}{r(\mathbf{k})} \cdot |S_N(e_{\mathbf{k}}, \omega)| \right)$$

$r(\mathbf{k}) = \prod_{i=1}^{s} \max\{1, |k_i|\}$.
$\Delta^*(M) = \{\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}\} : ||\mathbf{k}||_\infty = \max_{1\leq i\leq s} |k_i| < M\}$,
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Important types of sequences

Type 1: Kronecker sequences

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$:

$$\omega = (\{n\boldsymbol{\alpha}\})_{n\geq 0}\,, \quad \text{with suitable rational or irrational } \boldsymbol{\alpha}$$

$$\mathcal{T}^{(s)} = \{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^s\}, \quad \text{the trigonometric function system on } [0,1[^s$$

Type 2: digital sequences

$b \geq 2$ integer:

$$\omega = \text{a digital } (t,m,s)\text{-sequence (or -net) in base } b,$$

$$\mathcal{W}_b^{(s)} = \{w_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}, \quad \text{the Walsh function system in base } b \text{ on } [0,1[^s.$$

digital sequences in base b:

$\omega = ((C_1 \vec{n}^T, \ldots, C_s \vec{n}^T))_{n\geq 0}$, with suitably chosen matrices $C_1, \ldots, C_s$

Remark

Some (digital) sequences are closely related to the p-adic integers ...

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The p-adic integers

Definition

The compact abelian group of p-adic integers $\mathbb{Z}_p$:

$$\mathbb{Z}_p = \left\{ z = \sum_{j \geq 0} z_j\, p^j, \quad \text{with digits } z_j \in \{0, 1, \ldots, p-1\} \right\},$$

Remark

$\mathbb{Z}$ is embedded in $\mathbb{Z}_p$.

Definition

The dual group $\hat{\mathbb{Z}}_p$:

$$\hat{\mathbb{Z}}_p = \{\chi_0\} \cup \{\underbrace{z \mapsto e^{2\pi i \frac{a}{p^g} z}}_{\chi(a,g;z)} : 0 < a < p^g, a, g \in \mathbb{N}\},$$

χ_0 ...the trivial character, $\chi_0(z) = 1\ \forall z \in \mathbb{Z}_p$

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p-adic Monna's map

Definition (Monna's map)

$$\varphi_p : \quad \mathbb{Z}_p \to [0,1[,$$
$$\varphi_p\Big(\sum_{j\geq 0} z_j\, p^j\Big) = \sum_{j\geq 0} z_j\, p^{-j-1} \pmod 1.$$

Properties

- φ_p is continuous and surjective, but *not injective*.
- φ_p gives a bijection between $\mathbb{N} \subset \mathbb{Z}_p$ and the set of reduced p-adic fractions

$$\{a/p^g : 0 < a < p^g,\ a, g \in \mathbb{N}, (a, p^g) = (a, p) = 1\},$$

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$$\hat{\mathbb{Z}}_p = \{\underbrace{z \mapsto e^{2\pi i \varphi_p(k) z}}_{\chi_k(z)} : \ k \in \mathbb{N}_0\},$$

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$$\varphi_p^+ : \quad [0,1[\to \mathbb{Z}_p,$$
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$$\text{if } x = \sum_{j \geq 0} x_j p^{-j-1} \text{ with } x_j \neq p-1 \text{ infinitely often}$$

Definition (The p-adic function system in dimension $s = 1$)

$k \in \mathbb{N}_0$

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Let $\Gamma_p = \{\gamma_k : k \in \mathbb{N}_0\}$.

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$\mathbf{p} = (p_1, \ldots, p_s)$ a vector of s not necessarily distinct primes p_i

$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$

$$\gamma_{\mathbf{k}} : \ [0,1[^s \to \{c \in \mathbb{C} : \ |c| = 1\},$$

$$\gamma_{\mathbf{k}}(\mathbf{x}) \ = \ \prod_{i=1}^{s} \gamma_{k_i}(x_i), \quad \gamma_{k_i} \in \Gamma_{p_i}.$$

ONB property

$\Gamma_{\mathbf{p}}^{(s)} = \{\gamma_{\mathbf{k}} : \ \mathbf{k} \in \mathbb{N}_0^s\}$ is an orthonormal basis of $L^2([0,1[^s)$.

Fourier coefficients

f integrable on $[0,1[^s$:

$$\hat{f}(\mathbf{k}) \ = \ \int_{[0,1[^s} f(\mathbf{x})\overline{\gamma_{\mathbf{k}}(\mathbf{x})} \, d\mathbf{x}.$$

p-adic Weyl criterion

Theorem (p-adic Weyl Criterion)

$\mathbf{p} = (p_1, \dots, p_s)$ *a vector of s not necessarily distinct primes* p_i

$\omega = (\mathbf{x}_n)_{n \geq 0}$ *uniformly distributed in* $[0, 1[^s$ *if and only if*

$$\lim_{N \to \infty} S_N(\gamma_{\mathbf{k}}, \omega) = 0 \quad \forall \mathbf{k} \neq \mathbf{0}.$$

Corollary1

$\omega = (x_n)_{n \geq 0}, \; x_n = \varphi_p(n)$ ... van der Corput sequence in prime base p.

Then ω is uniformly distributed modulo one.

Corollary2

$\omega = (\mathbf{x}_n)_{n \geq 0}, \; \mathbf{x}_n = (\varphi_{p_1}(n), \dots, \varphi_{p_s}(n))$ ... Halton sequence (i.e., coprime bases p_i).

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p-adic Inequality of Erdös-Turán-Koksma

Theorem

ω a sequence in $[0,1[^s$, $g \in \mathbb{N}$ arbitrary,
for the ease of notation: $\mathbf{p} = (p, \ldots, p)$, p prime

$$D_N(\omega) \;\le\; \underbrace{1-(1-2/p^g)^s}_{\text{discretization error}} + \sum_{\mathbf{k}\in\Delta^*(g)} \rho(\mathbf{k})\,|S_N(\gamma_{\mathbf{k}},\omega)| \;,$$

where

$$\rho(k) \;=\; \begin{cases} 1 & \text{if} \quad k=0 \\ \dfrac{2}{p^t \sin(\pi k_{t-1}/p)} & \text{if} \quad p^{t-1} \le k < p^t,\; t \in \mathbb{N}, \end{cases}$$

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Main Lemma

Question

Where do the weights $\rho(k)$ come from?

Lemma

Let $f(\mathbf{x}) = \mathbf{1}_I(\mathbf{x}) - \lambda(I)$,
where $I = \prod_{i=1}^{s} \left[a_i p^{-g}, b_i p^{-g}\right[, \quad 0 \le a_i < b_i \le p^g,\ g \ge 1.$

Then

1. $\forall\, \mathbf{k} \in \mathbb{N}_0^s \setminus \Delta^*(g)$,
$$\hat{f}(\mathbf{k}) = 0,$$
2. $\forall\, \mathbf{k} \in \Delta^*(g)$,
$$|\hat{\mathbf{1}}_I(\mathbf{k})| \le \rho(\mathbf{k}),$$

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Let $f(\mathbf{x}) = \mathbf{1}_I(\mathbf{x}) - \lambda(I)$,
where $I = \prod_{i=1}^{s} \left[a_i p^{-g}, b_i p^{-g}\right[, \quad 0 \leq a_i < b_i \leq p^g,\ g \geq 1.$

Then

1. $\forall\, \mathbf{k} \in \mathbb{N}_0^s \setminus \Delta^*(g)$,
$$\hat{f}(\mathbf{k}) = 0,$$
2. $\forall\, \mathbf{k} \in \Delta^*(g)$,
$$|\hat{\mathbf{1}}_I(\mathbf{k})| \leq \rho(\mathbf{k}),$$

Main Lemma

Question

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$\mathbf{p}$-adic Diaphony

Definition ($\mathbf{p}$-adic Diaphony)

$\mathbf{p} = (p_1, \ldots, p_s)$ a vector of s not necessarily distinct primes p_i
ω a sequence in $[0, 1[^s$

$$F_N(\omega) = \left(\frac{1}{\sigma - 1} \sum_{\mathbf{k} \neq \mathbf{0}} \rho_{\mathbf{p}}(\mathbf{k}) \left| S_N(\gamma_{\mathbf{k}}, \omega) \right|^2 \right)^{1/2}$$

where

$$\rho_{p_i}(k_i) = \begin{cases} 1 & \text{if} \quad k = 0 \\ \dfrac{1}{p_i^{2t_i - 2}} & \text{if} \quad p_i^{t_i - 1} \leq k < p_i^{t_i}, \; t_i \in \mathbb{N}, \end{cases}$$

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Results I

Theorem

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1. $0 \leq F_N(\omega) \leq 1$.
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Theorem (Regular grid)

$\mathbf{p} = (p, \ldots, p)$, *p prime*
$g \in \mathbb{N}$,
$\omega = (1/p^g)\mathbb{Z}^s \pmod 1$ *the regular grid of* p^{gs} *points in* $[0,1[^s$.
Put $N = p^{gs}$.
Then

$$C_1 \cdot \frac{1}{N^{1/s}} \leq F_N(\omega) \leq C_2 \cdot \frac{\log N}{N^{1/s}}, \quad C_1, C_2 \text{ constants}$$

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Theorem (Inequality of Erdös-Turán-Koksma)

ω a sequence in $[0,1[^s$,
$\mathbf{g} \in \mathbb{N}^s$ arbitrary

$$F_N^2(\omega) \leq \frac{\sigma}{\sigma-1} s\delta + \frac{1}{\sigma-1} \sum_{\mathbf{k} \in \Delta_{\mathbf{p}}^*(\mathbf{g})} \rho_{\mathbf{p}}(\mathbf{k}) \left| S_N(\gamma_{\mathbf{k}}, \omega) \right|^2 ,$$

where

$$\Delta_{\mathbf{p}}(\mathbf{g}) = \left\{ \mathbf{k} = (k_1, \ldots, k_s) : \; 0 \leq k_i < p_i^{g_i}, 1 \leq i \leq s \right\},$$
$$\Delta_{\mathbf{p}}^*(\mathbf{g}) = \Delta_{\mathbf{p}}(\mathbf{g}) \setminus \{\mathbf{0}\},$$
$$\delta = \max_i (p_i + 1)^{-1} p_i^{-g_i+1}.$$

Corollary

Weyl's Criterion follows easily.

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Hybrid sequences revisited

Definition

Hybrid sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ in $[0,1[^s$:
mix at least two lower-dimensional sequences of different types s.t. certain coordinates of $\mathbf{x}_n$ stem from the first sequence, certain of the remaining coordinates stem from the second sequence, and so on.

Problem

Need new deterministic tools to analyze hybrid sequences.

Existing tools

Niederreiter (2010): inequality of ETK for $\mathcal{T}^{(s)}, \mathcal{W}_{\mathbf{b}}^{(s)}$

Situation

We have three different function systems on $[0,1[^s$ at our disposition:

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Hybrid function systems

Notation

$s \geq 2$:

$$s = s_1 + s_2 + s_3, \quad s_i \in \mathbb{N}_0$$

$\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s$:

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with $\mathbf{y}^{(1)} = (y_1, \ldots, y_{s_1})$, $\mathbf{y}^{(2)} = (y_{s_1+1}, \ldots, y_{s_1+s_2})$, and so on.

(More general selection principle possible: $\{1, \ldots, s\} = \mathbf{u}_1 \cup \mathbf{u}_2 \cup \mathbf{u}_3$ (disjoint))

Definition (Hybrid function system)

Let $s = s_1 + s_2 + s_3$, $s_i \in \mathbb{N}_0$
$\mathcal{T}^{(s_1)}$ the trigonometric function system in dimension s_1,
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Hybrid Weyl criterion

Theorem (Hybrid Weyl criterion)

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$\mathbf{b} = (b_1, \ldots, b_{s_2})$ *a vector of* s_2 *not necessarily distinct integers* $b_i \geq 2$,
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Then
$\omega = (\mathbf{x}_n)_{n\geq 0}$ *uniformly distributed in* $[0,1[^s$ *if and only if*

$$\lim_{N\to\infty} S_N(\xi_{\mathbf{k}}, \omega) = 0 \quad \forall \mathbf{k} \neq \mathbf{0}.$$

Proof.

Let ω be uniformly distributed in $[0,1[^s$. Each function $\xi_{\mathbf{k}}$ is Riemann-integrable, hence $\lim_{N\to\infty} S_N(\xi_{\mathbf{k}}, \omega) = 0$, for all $\mathbf{k} \neq 0$.
Let $\lim_{N\to\infty} S_N(\xi_{\mathbf{k}}, \omega) = 0$, for all $\mathbf{k} \neq 0$. We show that $\lim_{N\to\infty} S_N(\mathbf{1}_J - \lambda_s(J), \omega) = 0$, for appropriate intervals J in $[0,1[^s$, by a technique based on results from Niederreiter (2009) for Fourier analysis in terms of $\mathcal{T}^{(s_1)}$ and Hellekalek (1994, 2009) for $\mathcal{W}_{\mathbf{b}}^{(s_2)}$ and $\Gamma_{\mathbf{p}}^{(s_3)}$. □

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some weight functions

Weight functions

We define two types of weight functions:

$$k \in \mathbb{Z}: \qquad r(k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ k^{-2} & \text{if} \quad k \neq 0, \end{cases}$$

$$\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s: \qquad r(\mathbf{k}) = \prod_{i=1}^{s} r(k_i),$$

$$\sigma(r, s) = \sum_{\mathbf{k}} r(\mathbf{k}) = (1 + \pi^2/3)^s.$$

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$\Delta^* = \Delta \setminus \{\mathbf{0}\}$,

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Weyl's Criterion follows easily.

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Ongoing research

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- $\Gamma_{\mathbf{p}}^{(s)}$: generalization from $\mathbf{p}$ to general bases $\mathbf{b} = (b_1, \ldots, b_s)$, $b_i \geq 2$
- hybrid inequality of Le Veque and inequality of ETK
- joint paper with Harald Niederreiter on LDS:
  study various sequences and point sets $\omega = (\mathbf{x}_n)_{n \geq 0}$, for example of the form

  $$\mathbf{x}_n = (\varphi_{p_1}(f_1(n)), \ldots, \varphi_{p_s}(f_s(n))), \quad n \geq 0,$$

  where $f_i : \mathbb{N}_0 \to \mathbb{N}_0,\ 1 \leq i \leq s$.
- double sum representation of hybrid diaphony:
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$$F_N^2(\omega) = \frac{1}{\sigma - 1} \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \phi(\mathbf{x}_n \dot{-} \mathbf{x}_m).$$

Thank your for your attention!

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In preparation

- hybrid spectral test
- $\Gamma_{\mathbf{p}}^{(s)}$: generalization from $\mathbf{p}$ to general bases $\mathbf{b} = (b_1, \dots, b_s)$, $b_i \geq 2$
- hybrid inequality of Le Veque and inequality of ETK
- joint paper with Harald Niederreiter on LDS:
  study various sequences and point sets $\omega = (\mathbf{x}_n)_{n \geq 0}$, for example of the form

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