Pointwise Approximation of Stochastic Heat Equations with Additive Noise

Daniel Henkel

TU Darmstadt
Partially supported by the DFG

Contents

Strong Approximation of SPDEs

- *Grecksch, Kloeden (1996)*, *Gyöngy, Nualart (1997)* and **. . .** :
  Upper bounds for error of algorithms based on uniform discretization in space and time.

- *Davie, Gaines (2001)* and *Müller-Gronbach, Ritter (2007)*:
  Lower bounds for classes of algorithms, optimality.
  Non-uniform time discretization.

Stochastic Heat Equation with Additive Noise

$$\begin{aligned} dX(t) &= \Delta X(t)\,dt + B(t)\,dW(t), \quad t \in (0, T], \\ X(0) &= 0, \end{aligned}$$

with Dirichlet boundary conditions

Stochastic Heat Equation with Additive Noise

$$\begin{aligned} dX(t) &= \Delta X(t)\, dt + B(t)\, dW(t), \quad t \in (0, T], \\ X(0) &= 0, \end{aligned}$$

with Dirichlet boundary conditions and

- $W(t) = \sum_{i \in \mathbb{N}^d} |i|_2^{-\frac{\gamma}{2}} \cdot \beta_i(t) \cdot h_i$
  Brownian motion in $H = L_2((0,1)^d)$ in which
  - (β_i) independent standard one-dimensional Brownian motions,
  - $h_i(u) = 2^{\frac{d}{2}} \prod_{k=1}^{d} \sin(i_k \pi u_k)$ eigenfunction of Δ for $i \in \mathbb{N}^d$,

Stochastic Heat Equation with Additive Noise

$$\begin{aligned} dX(t) &= \Delta X(t)\,dt + B(t)\,dW(t), \quad t \in (0, T], \\ X(0) &= 0, \end{aligned}$$

with Dirichlet boundary conditions and

- $W(t) = \sum_{i \in \mathbb{N}^d} |i|_2^{-\frac{\gamma}{2}} \cdot \beta_i(t) \cdot h_i$
  Brownian motion in $H = L_2((0,1)^d)$ in which
  - (β_i) independent standard one-dimensional Brownian motions,
  - $h_i(u) = 2^{\frac{d}{2}} \prod_{k=1}^{d} \sin(i_k \pi u_k)$ eigenfunction of Δ for $i \in \mathbb{N}^d$,
- B operator-valued mapping.

Case 1 (TC(γ)) $\gamma > d$.
Case 2 (ID) $\gamma = 0$ and $d = 1$.

Computational Problem

Task

Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions β_i's at a finite number of points.

Computational Problem

Task

Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions β_i's at a finite number of points.

Error and cost of any approximation $\widehat{X}(T)$

$$
\begin{aligned}
\mathrm{e}\left(\widehat{X}(T)\right) &= \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2}, \\
\mathrm{cost}\left(\widehat{X}(T)\right) &= \text{total number of evaluations of the } \beta_i\text{'s.}
\end{aligned}
$$

Computational Problem

Task

Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions β_i's at a finite number of points.

Error and cost of any approximation $\widehat{X}(T)$

$$\begin{aligned} \mathrm{e}\left(\widehat{X}(T)\right) &= \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2}, \\ \mathrm{cost}\left(\widehat{X}(T)\right) &= \text{total number of evaluations of the } \beta_i\text{'s.} \end{aligned}$$

Goal

Approximation with optimal relation between error and cost.

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_{\mathrm{uni}}$ of algorithms with **uniform** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_k = \frac{k}{n} T$, $k = 1, \ldots, n$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}_{\text{uni}}$ of algorithms with **uniform** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- use nodes $t_k = \frac{k}{n} T$, $k = 1, \ldots, n$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Approximation:

$$\widehat{X}(T) = \phi(\beta_{i_1}(t_1), \ldots, \beta_{i_1}(t_n), \ldots, \beta_{i_\ell}(t_1), \ldots, \beta_{i_\ell}(t_n))$$

for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}$ of algorithms with **arbitrary** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Classes of Algorithms and Minimal Errors

Class $\mathfrak{X}$ of algorithms with **arbitrary** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n_i \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\widehat{X}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n_{i_1},i_1}), \ldots, \beta_{i_\ell}(t_{1,i_\ell}), \ldots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}(T)\right) = N.$$

Classes of Algorithms and Minimal Errors

N-th minimal errors

$$\begin{aligned} e_{\mathrm{uni}}(N) &= \inf\left\{ e\left(\widehat{X}(T)\right) : \widehat{X}(T) \in \mathfrak{X}_{\mathrm{uni}} \text{ and } \operatorname{cost}\left(\widehat{X}(T)\right) \leq N \right\}, \\ e(N) &= \inf\left\{ e\left(\widehat{X}(T)\right) : \widehat{X}(T) \in \mathfrak{X} \quad \text{ and } \operatorname{cost}\left(\widehat{X}(T)\right) \leq N \right\}. \end{aligned}$$

Clearly

$$e(N) \leq e_{\mathrm{uni}}(N).$$

Classes of Algorithms and Minimal Errors

N-th minimal errors

$$\begin{aligned} e_{uni}(N) &= \inf\left\{ e\left(\widehat{X}(T)\right) : \widehat{X}(T) \in \mathfrak{X}_{uni} \text{ and } \operatorname{cost}\left(\widehat{X}(T)\right) \le N \right\}, \\ e(N) &= \inf\left\{ e\left(\widehat{X}(T)\right) : \widehat{X}(T) \in \mathfrak{X} \quad \text{ and } \operatorname{cost}\left(\widehat{X}(T)\right) \le N \right\}. \end{aligned}$$

Clearly

$$e(N) \le e_{uni}(N).$$

Questions:

- Rate of convergence of $e(N)$ and $e_{uni}(N)$?
  Superiority of $\mathfrak{X}$ over $\mathfrak{X}_{uni}$?
- Construction of algorithms $\widehat{X}_N(T) \in \mathfrak{X}$ with $\operatorname{cost}\left(\widehat{X}_N(T)\right) \le N$ and $e\left(\widehat{X}_N(T)\right) \asymp e(N)$.
  Likewise for $\mathfrak{X}_{uni}$.

Results for Equations with Additive Noise

Assumption (A(α)): Put $B_{i,j}(t) = \langle B(t)h_i, h_j \rangle$.

- For $d = 1$ and $\alpha > 1$ assume

$$\sup_{t\in[0,T]} \left((B_{i,j}(t))^2 + (B'_{i,j}(t))^2\right) \preceq \frac{1}{|i-j|^\alpha + 1} \qquad (1)$$

$$\text{and} \qquad \inf_{t\in[0,T]} (B_{i,i}(t))^2 > 0. \qquad (2)$$

Results for Equations with Additive Noise

Assumption (A(α)): Put $B_{i,j}(t) = \langle B(t)h_i, h_j \rangle$.

- For $d = 1$ and $\alpha > 1$ assume

$$\sup_{t\in[0,T]} \left((B_{i,j}(t))^2 + (B'_{i,j}(t))^2\right) \preceq \frac{1}{|i-j|^\alpha + 1} \qquad (1)$$

$$\text{and} \qquad \inf_{t\in[0,T]} (B_{i,i}(t))^2 > 0. \qquad (2)$$

Example: If $\alpha = 2$, (1) holds for multiplication operators $B(t)$, i.e.

$$(B(t)h)(u) = G(t,u) \cdot h(u)$$

with $G \in C^{(1,1)}([0,T] \times [0,1])$.

Results for Equations with Additive Noise

Assumption (A(α)): Put $B_{i,j}(t) = \langle B(t)h_i, h_j\rangle$.

- For $d = 1$ and $\alpha > 1$ assume

$$\sup_{t\in[0,T]} \left((B_{i,j}(t))^2 + (B'_{i,j}(t))^2\right) \preceq \frac{1}{|i-j|^\alpha + 1} \tag{1}$$

$$\text{and} \qquad \inf_{t\in[0,T]} (B_{i,i}(t))^2 > 0. \tag{2}$$

- For $d \geq 2$ and $\alpha > d$ assume

$$\sup_{t\in[0,T]} \left((B_{i,j}(t))^2 + (B'_{i,j}(t))^2\right) \preceq \prod_{k=1}^{d} \frac{1}{|i_k - j_k|^\alpha + 1}$$

$$\text{and} \qquad \inf_{t\in[0,T]} (B_{i,i}(t))^2 > 0.$$

Results for Equations with Additive Noise

Theorem *Henkel* (2009)
Assume (ID) and (A(α)). Then

$$N^{-1/6} \preceq e_{\mathrm{uni}}(N) \preceq \begin{cases} N^{-(\alpha-1)/6}, & \text{if } 1 < \alpha < 2, \\ N^{-1/6}, & \text{if } 2 \le \alpha < \infty, \end{cases}$$

and

$$N^{-1/2} \preceq e(N) \preceq \begin{cases} N^{-(\alpha-1)/4}, & \text{if } 1 < \alpha < 2, \\ N^{-1/4}, & \text{if } 2 \le \alpha < \infty. \end{cases}$$

Results for Equations with Additive Noise

Theorem *Henkel* (2009)
Assume (ID) and (A(α)). Then

$$N^{-1/6} \preceq e_{\text{uni}}(N) \preceq \begin{cases} N^{-(\alpha-1)/6}, & \text{if } 1 < \alpha < 2, \\ N^{-1/6}, & \text{if } 2 \le \alpha < \infty, \end{cases}$$

and

$$N^{-1/2} \preceq e(N) \preceq \begin{cases} N^{-(\alpha-1)/4}, & \text{if } 1 < \alpha < 2, \\ N^{-1/4}, & \text{if } 2 \le \alpha < \infty. \end{cases}$$

Remarks

- Suboptimality of $\mathfrak{X}_{\text{uni}}$ (at least), if $\alpha > \frac{5}{3}$.
- Limiting case $\alpha \to \infty$, i.e. $B(t) = id$:

$$e_{\text{uni}}(N) \asymp N^{-1/6}$$

and

$$e(N) \asymp N^{-1/2},$$

see *Müller-Gronbach, Ritter, Wagner* (2008).

Results for Equations with Additive Noise

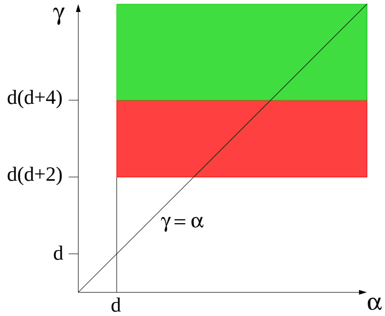
Theorem *Henkel* (2010)

Assume (TC(γ)) and (A(α)). Let $\epsilon > 0$. Then for $\gamma > d(d+2)$

$$N^{-\frac{2}{d+2}} \preceq e_{\mathrm{uni}}(N) \preceq \begin{cases} N^{-\frac{\gamma-(d(d+2))}{d(d+2)}+\epsilon}, & \text{if } \gamma < d(d+4), \\ N^{-\frac{2}{d+2}+\epsilon}, & \text{if } \gamma \geq d(d+4), \end{cases}$$

and

$$N^{-1} \preceq e(N) \preceq N^{-\frac{d+2}{4d}+\epsilon}, \quad \text{if } \gamma \leq \alpha \text{ and } d \geq 3.$$



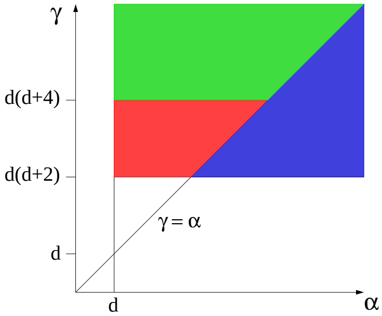
Results for Equations with Additive Noise

Theorem *Henkel* (2010)
Assume (TC(γ)) and (A(α)). Let $\epsilon > 0$. Then for $\gamma > d(d+2)$

$$N^{-\frac{2}{d+2}} \preceq e_{\mathrm{uni}}(N) \preceq \begin{cases} N^{-\frac{\gamma-(d(d+2))}{d(d+2)}+\epsilon}, & \text{if } \gamma < d(d+4), \\ N^{-\frac{2}{d+2}+\epsilon}, & \text{if } \gamma \geq d(d+4), \end{cases}$$

and

$$N^{-1} \preceq e(N) \preceq N^{-\frac{d+2}{4d}+\epsilon}, \quad \text{if } \gamma \leq \alpha \text{ and } d \geq 3.$$



Results for Equations with Additive Noise

Theorem *Henkel* (2010)
Assume (TC(γ)) and (A(α)). Let $\epsilon > 0$. Then for $\gamma > d(d+2)$

$$N^{-\frac{2}{d+2}} \preceq \mathrm{e}_{\mathrm{uni}}(N) \preceq \begin{cases} N^{-\frac{\gamma-(d(d+2))}{d(d+2)}+\epsilon}, & \text{if } \gamma < d(d+4), \\ N^{-\frac{2}{d+2}+\epsilon}, & \text{if } \gamma \geq d(d+4), \end{cases}$$

and

$$N^{-1} \preceq \mathrm{e}(N) \preceq N^{-\frac{d+2}{4d}+\epsilon}, \quad \text{if } \gamma \leq \alpha \text{ and } d \geq 3.$$

Remarks

- Suboptimality of $\mathfrak{X}_{\mathrm{uni}}$ (at least), if $d \geq 3$ and $\alpha \geq \gamma > d(d+2)$.
- Limiting case $\alpha \to \infty$, i.e. $B(t) = id$:

$$\mathrm{e}(N) \asymp \begin{cases} N^{-\frac{\gamma+2-d}{2d}}, & \text{if } \gamma < 3d-2, \\ N^{-1}, & \text{if } \gamma > 3d-2. \end{cases}$$

see *Müller-Gronbach, Ritter, Wagner* (2008).

Results for Equations with Additive Noise

Remark Upper bounds for e(N)

- Time discretizations are quantiles of the density $t \mapsto \exp(-\frac{\mu_j}{3}(T-t))$, i.e.

  $$\int_0^{s_{k,j}} \exp\left(-\frac{\mu_j}{3}(T-t)\right)\, dt = \frac{k}{\nu_j} \int_0^T \exp\left(-\frac{\mu_j}{3}(T-t)\right)\, dt$$

  for $j \in \mathcal{J} \subset \mathbb{N}^d$, $\mu_j = \pi^2 |j|_2^2$, $\nu_j \in \mathbb{N}$, $k = 1, \ldots, \nu_j$ and $\{t_{1,i}, \ldots, t_{n,i}\} = \bigcup_{j \in \mathcal{J}} \{s_{1,j}, \ldots, s_{\nu_j,j}\}$ for every β_i.
- Drift-implicit Euler-Maruyama scheme.

Summary and Outlook

Summary

For additive noise with decay condition (A(α)) the minimal error $e(N)$ is superior to $\mathrm{e}_{\mathrm{uni}}(N)$, if

(ID) $\alpha > \frac{5}{3}$.

(TC(γ)) $\alpha \geq \gamma > d(d+2)$ and $d \geq 3$.

Summary and Outlook

Summary

For additive noise with decay condition (A(α)) the minimal error $e(N)$ is superior to $e_{\mathrm{uni}}(N)$, if

(ID) $\alpha > \frac{5}{3}$.

(TC(γ)) $\alpha \geq \gamma > d(d+2)$ and $d \geq 3$.

Outlook

- The cases $\gamma \leq d(d+2)$ and $d < 3$.
- Sharp bounds for $e(N)$.
- Multiplicative noise.