Pointwise Approximation of Stochastic Heat Equations with Additive Noise

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Strong Approximation of SPDEs

- **Grecksch, Kloeden (1996), Gyöngy, Nualart (1997) and ...**: Upper bounds for error of algorithms based on uniform discretization in space and time.

Stochastic Heat Equation with Additive Noise

\[ dX(t) = \Delta X(t) \, dt + B(t) \, dW(t), \quad t \in (0, T], \]
\[ X(0) = 0, \]

with Dirichlet boundary conditions
Stochastic Heat Equation with Additive Noise

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with Dirichlet boundary conditions and

\[ W(t) = \sum_{i \in \mathbb{N}^d} |i|^{-\frac{\gamma}{2}} \cdot \beta_i(t) \cdot h_i \]

Brownian motion in \( H = L_2((0,1)^d) \) in which

\[ (\beta_i) \text{ independent standard one-dimensional Brownian motions}, \]
\[ h_i(u) = 2^{\frac{d}{2}} \prod_{k=1}^{d} \sin(i_k \pi u_k) \text{ eigenfunction of } \Delta \text{ for } i \in \mathbb{N}^d, \]
Stochastic Heat Equation with Additive Noise

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  Brownian motion in \( H = L_2((0, 1)^d) \) in which
    - \((\beta_i)\) independent standard one-dimensional Brownian motions,
    - \( h_i(u) = 2^{\frac{d}{2}} \prod_{k=1}^{d} \sin(i_k \pi u_k) \) eigenfunction of \( \Delta \) for \( i \in \mathbb{N}^d \),
- \( B \) operator-valued mapping.

**Case 1** (TC(\(\gamma\))) \(\gamma > d\).

**Case 2** (ID) \(\gamma = 0\) and \(d = 1\).
Computational Problem

Task
Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions $\beta_i$’s at a finite number of points.
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Error and cost of any approximation $\hat{X}(T)$

$$e\left(\hat{X}(T)\right) = \left(\mathbb{E}\|X(T) - \hat{X}(T)\|_H^2\right)^{1/2},$$
$$\text{cost}\left(\hat{X}(T)\right) = \text{total number of evaluations of the } \beta_i \text{'s.}$$
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Goal
Approximation with optimal relation between error and cost.
Classes of Algorithms and Minimal Errors

Class $\mathcal{X}_{\text{uni}}$ of algorithms with \textbf{uniform} time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate $\beta_i$ with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for $\beta_i$ with $i \in \mathcal{I}$.
- use nodes $t_k = \frac{k}{n} T$, $k = 1, \ldots, n$, for $\beta_i$ with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \rightarrow H$ measurable with $N = |\mathcal{I}| \cdot n$. 
Classes of Algorithms and Minimal Errors

Class $\mathcal{X}_{\text{uni}}$ of algorithms with **uniform** time discretization:

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- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Approximation:

$$\hat{X}(T) = \phi(\beta_{i_1}(t_1), \ldots, \beta_{i_1}(t_n), \ldots, \beta_{i_\ell}(t_1), \ldots, \beta_{i_\ell}(t_n))$$

for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.

Clearly

$$\text{cost} \left( \hat{X}(T) \right) = N.$$
Classes of Algorithms and Minimal Errors

Class $\mathcal{X}$ of algorithms with **arbitrary** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate $\beta_i$ with $i \in \mathcal{I}$.
  - choose $n_i \in \mathbb{N}$: number of evaluations for $\beta_i$ with $i \in \mathcal{I}$.
  - choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for $\beta_i$ with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \rightarrow H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$. 
Classes of Algorithms and Minimal Errors

Class $\mathcal{X}$ of algorithms with \textbf{arbitrary} time discretization:
- choose finite set $\mathcal{I} \subset \mathbb{N}^d$: only evaluate $\beta_i$ with $i \in \mathcal{I}$.
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- choose $\phi : \mathbb{R}^N \rightarrow H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\hat{\mathcal{X}}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n_{i_1},i_1}), \ldots, \beta_{i_\ell}(t_{1,i_\ell}), \ldots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

for $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.

Clearly

$$\text{cost} \left( \hat{\mathcal{X}}(T) \right) = N.$$
Classes of Algorithms and Minimal Errors

$N$-th minimal errors

\[
e_{\text{uni}}(N) = \inf \left\{ e(\hat{X}(T)) : \hat{X}(T) \in \mathcal{X}_{\text{uni}} \text{ and cost } (\hat{X}(T)) \leq N \right\},
\]

\[
e(N) = \inf \left\{ e(\hat{X}(T)) : \hat{X}(T) \in \mathcal{X} \text{ and cost } (\hat{X}(T)) \leq N \right\}.
\]

Clearly

\[
e(N) \leq e_{\text{uni}}(N).
\]
Classes of Algorithms and Minimal Errors

$N$-th minimal errors

\[
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Clearly

\[
e(N) \leq e_{\text{uni}}(N).
\]

Questions:

- Rate of convergence of $e(N)$ and $e_{\text{uni}}(N)$?
- Superiority of $\mathcal{X}$ over $\mathcal{X}_{\text{uni}}$?
- Construction of algorithms $\hat{X}_N(T) \in \mathcal{X}$ with $\text{cost}\left(\hat{X}_N(T)\right) \leq N$ and $e\left(\hat{X}_N(T)\right) \simeq e(N)$. Likewise for $\mathcal{X}_{\text{uni}}$. 
Results for Equations with Additive Noise

**Assumption (A($\alpha$)):** Put $B_{i,j}(t) = \langle B(t)h_i, h_j \rangle$.

- For $d = 1$ and $\alpha > 1$ assume

\[
\sup_{t \in [0, T]} \left( (B_{i,j}(t))^2 + (B'_{i,j}(t))^2 \right) \leq \frac{1}{|i - j|^{\alpha + 1}} \tag{1}
\]

and

\[
\inf_{t \in [0, T]} (B_{i,i}(t))^2 > 0. \tag{2}
\]
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  $$
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  $$

  and

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  \inf_{t \in [0,T]} (B_{i,i}(t))^2 > 0. \quad (2)
  $$

**Example:** If $\alpha = 2$, (1) holds for multiplication operators $B(t)$, i.e.

$$
(B(t)h)(u) = G(t, u) \cdot h(u)
$$

with $G \in C^{(1,1)}([0, T] \times [0, 1])$. 
Assumption (A(\(\alpha\))): \(B_{i,j}(t) = \langle B(t)h_i, h_j \rangle\).

- For \(d = 1\) and \(\alpha > 1\) assume
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  and
  \[
  \inf_{t \in [0,T]} (B_{i,i}(t))^2 > 0. \tag{2}
  \]

- For \(d \geq 2\) and \(\alpha > d\) assume
  \[
  \sup_{t \in [0,T]} \left( (B_{i,j}(t))^2 + (B'_{i,j}(t))^2 \right) \leq \prod_{k=1}^{d} \frac{1}{|i_k-j_k|^{\alpha+1}} \tag{3}
  \]
  and
  \[
  \inf_{t \in [0,T]} (B_{i,i}(t))^2 > 0. \tag{4}
  \]
Results for Equations with Additive Noise

**Theorem** Henkel (2009)
Assume (ID) and (A(\(\alpha\))). Then

\[
N^{-1/6} \lesssim e_{\text{uni}}(N) \preceq \begin{cases} 
N^{-(\alpha-1)/6}, & \text{if } 1 < \alpha < 2, \\
N^{-1/6}, & \text{if } 2 \leq \alpha < \infty,
\end{cases}
\]

and

\[
N^{-1/2} \lesssim e(N) \preceq \begin{cases} 
N^{-(\alpha-1)/4}, & \text{if } 1 < \alpha < 2, \\
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**Remarks**

- Suboptimality of \(X_{\text{uni}}\) (at least), if \(\alpha > \frac{5}{3}\).
- Limiting case \(\alpha \to \infty\), i.e. \(B(t) = \text{id}\):

\[ e_{\text{uni}}(N) \preceq N^{-1/6} \]

and

\[ e(N) \preceq N^{-1/2}, \]

Results for Equations with Additive Noise

**Theorem** Henkel (2010)
Assume (TC(\(\gamma\))) and (A(\(\alpha\))). Let \(\epsilon > 0\). Then for \(\gamma > d(d + 2)\)

\[
N^{-\frac{2}{d+2}} \leq e_{uni}(N) \leq \left\{ \begin{array}{ll}
N^{-\frac{\gamma-(d(d+2))}{d(d+2)}} + \epsilon, & \text{if } \gamma < d(d + 4), \\
N^{-\frac{2}{d+2}} + \epsilon, & \text{if } \gamma \geq d(d + 4),
\end{array} \right.
\]

and

\[
N^{-1} \leq e(N) \leq N^{-\frac{d+2}{4d}} + \epsilon, \quad \text{if } \gamma \leq \alpha \text{ and } d \geq 3.
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**Remarks**

- Suboptimality of \(X_{uni}\) (at least), if \(d \geq 3\) and \(\alpha \geq \gamma > d(d + 2)\).
- Limiting case \(\alpha \to \infty\), i.e. \(B(t) = id\):

\[
e(N) \asymp \begin{cases} 
N^{-\frac{\gamma+2-d}{2d}}, & \text{if } \gamma < 3d - 2, \\
N^{-1}, & \text{if } \gamma > 3d - 2.
\end{cases}
\]

Remark Upper bounds for $e(N)$

- Time discretizations are quantiles of the density
  $t \mapsto \exp(-\frac{\mu_j}{3}(T - t))$, i.e.
  \[
  \int_{0}^{s_{k,j}} \exp\left(-\frac{\mu_j}{3}(T - t)\right) \, dt = \frac{k}{\nu_j} \int_{0}^{T} \exp\left(-\frac{\mu_j}{3}(T - t)\right) \, dt
  \]
  for $j \in J \subset \mathbb{N}^d$, $\mu_j = \pi^2|j|^2$, $\nu_j \in \mathbb{N}$, $k = 1, \ldots, \nu_j$ and
  $\{t_{1,i}, \ldots, t_{n,i}\} = \bigcup_{j \in J} \{s_{1,j}, \ldots, s_{\nu_j,j}\}$ for every $\beta_i$.

- Drift-implicit Euler-Maruyama scheme.
Summary

For additive noise with decay condition \( A(\alpha) \) the minimal error \( e(N) \) is superior to \( e_{\text{uni}}(N) \), if

\[
\text{(ID)} \quad \alpha > \frac{5}{3}.
\]

\[
\text{(TC}(\gamma)) \quad \alpha \geq \gamma > d(d + 2) \text{ and } d \geq 3.
\]
Summary and Outlook

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For additive noise with decay condition \((A(\alpha))\) the minimal error \(e(N)\) is superior to \(e_{\text{uni}}(N)\), if

\[(\text{ID}) \quad \alpha > \frac{5}{3}.\]

\[(\text{TC}(\gamma)) \quad \alpha \geq \gamma > d(d + 2) \text{ and } d \geq 3.\]

Outlook

- The cases \(\gamma \leq d(d + 2) \text{ and } d < 3.\)

- Sharp bounds for \(e(N)\).

- Multiplicative noise.