An intermediate bound on the star discrepancy

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Outline

Rank-1 lattice rules and the star discrepancy

Known bounds on the star discrepancy

An intermediate bound

Numerical results

Rank-1 lattice rules and the star discrepancy

To approximate the integral

$$I_d(f) = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

one may use a rank-1 lattice rule with n points, namely,

$$Q_{n,d}(f) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\left\{\frac{j\boldsymbol{z}}{n}\right\}\right),$$

where $\boldsymbol{z} \in \mathbb{Z}^d$ has no factor in common with n and the braces around a vector indicate that we take the fractional part of each component.

We denote the point set by $P_n(\boldsymbol{z})$.

We use the star discrepancy as a measure of goodness:

$$D^*(P_n(\boldsymbol{z})) := \sup_{\boldsymbol{x} \in [0,1)^d} \left| \mathrm{discr}(\boldsymbol{x}, P_n) \right|,$$

where $\mathrm{discr}(\boldsymbol{x}, P_n)$ is the ‘local discrepancy’ given by

$$\mathrm{discr}(\boldsymbol{x}, P_n) = \frac{|P_n(\boldsymbol{z}) \cap [\mathbf{0}, \boldsymbol{x})|}{n} - \mathrm{Vol}([\mathbf{0}, \boldsymbol{x})).$$

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Aside: The star discrepancy can be generalised to a weighted star discrepancy.

Known bounds on the star discrepancy

Existing algorithms to calculate $D^*(P_n(\boldsymbol{z}))$ have run times exponential in d.

In fact, an NP-hard problem. So computer searches to find a $\boldsymbol{z}$ with a low star discrepancy cannot use this quantity directly (unless d is small).

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So in calculations, we make use of computable bounds on the star discrepancy.

We set up some notation. Let $C(n) = \{h \in \mathbb{Z} : -n/2 < h \leq n/2\}$ and

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Then for $h \in C(n)$, let

$$r(h) = \max(1, |h|), \quad t(h, n) = \begin{cases} n \sin(\pi |h|/n) & \text{for } h \in C(n) \backslash 0, \\ 1 & \text{for } h = 0. \end{cases}$$

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Further, for $\boldsymbol{h} = (h_1, \ldots, h_d) \in C_d(n)$, set

$$r(\boldsymbol{h}) = \prod_{i=1}^{d} r(h_i) \quad \text{and} \quad t(\boldsymbol{h}, n) = \prod_{i=1}^{d} t(h_i, n).$$

With $H = \{\boldsymbol{h} \in C_d(n) \backslash \boldsymbol{0} : \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \mod n\}$, let

$$R(\boldsymbol{z}, n) = \sum_{\boldsymbol{h} \in H} \frac{1}{r(\boldsymbol{h})} \quad \text{and} \quad T(\boldsymbol{z}, n) = \sum_{\boldsymbol{h} \in H} \frac{1}{t(\boldsymbol{h}, n)}.$$

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Then

$$\begin{aligned} D^*(P_n(\boldsymbol{z})) &\leq 1 - \left(1 - \frac{1}{n}\right)^d + T(\boldsymbol{z}, n) \\ &\leq 1 - \left(1 - \frac{1}{n}\right)^d + \tfrac{1}{2} R(\boldsymbol{z}, n). \end{aligned}$$

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Though values of bound with $T(\boldsymbol{z}, n)$ are better than with $R(\boldsymbol{z}, n)/2$, calculation of $T(\boldsymbol{z}, n)$ requires $O(n^2 d)$ operations compared to $O(nd)$ for $R(\boldsymbol{z}, n)$.

An intermediate bound

We want to obtain a quantity $W(\boldsymbol{z}, n)$ between $T(\boldsymbol{z}, n)$ and $R(\boldsymbol{z}, n)/2$.

It should be such that:

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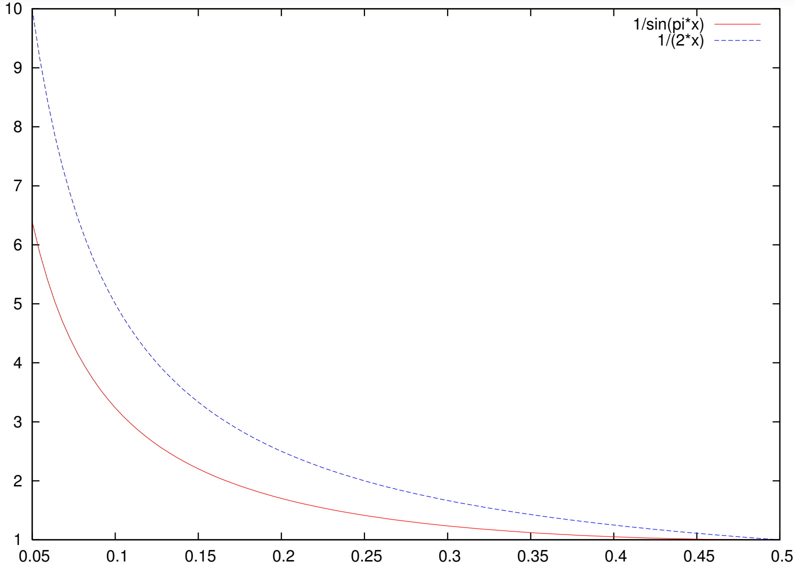
The reason why $T(\boldsymbol{z}, n) \leq R(\boldsymbol{z}, n)/2$ is essentially because

$$\frac{1}{\sin(\pi x)} \leq \frac{1}{2x}, \quad x \in (0, 1/2].$$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Suppose we can find a function G satisfying

$$\frac{1}{\sin(\pi x)} \le G(x) \le \frac{1}{2x} \quad \text{for } x \in (0, 1/2].$$

Then with

$$w(h, n) := \begin{cases} G(|h|/n)/n & \text{for } h \in C(n)\backslash 0, \\ 1 & \text{for } h = 0, \end{cases}$$

and $w(\boldsymbol{h}, n) = \prod_{i=1}^{d} w(h_i, n)$, the quantity

$$W(\boldsymbol{h}, n) = \sum_{\boldsymbol{h} \in H} w(\boldsymbol{h}, n)$$

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Asymptotic expansion used to calculate $R(\boldsymbol{z}, n)$ in $O(nd)$ operations can be modified for $W(\boldsymbol{z}, n)$ (assuming G is chosen appropriately).

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Laurent series of $1/\sin(\pi x)$ is

$$\frac{1}{\sin(\pi x)} = \frac{1}{\pi x} + \frac{\pi x}{6} + \frac{7\pi^3 x^3}{360} + \sum_{j=3}^{\infty} \frac{(-1)^{j+1} 2(2^{2j-1} - 1) B_{2j}}{(2j)!} (\pi x)^{2j-1},$$

where B_{2j} is a Bernoulli number.

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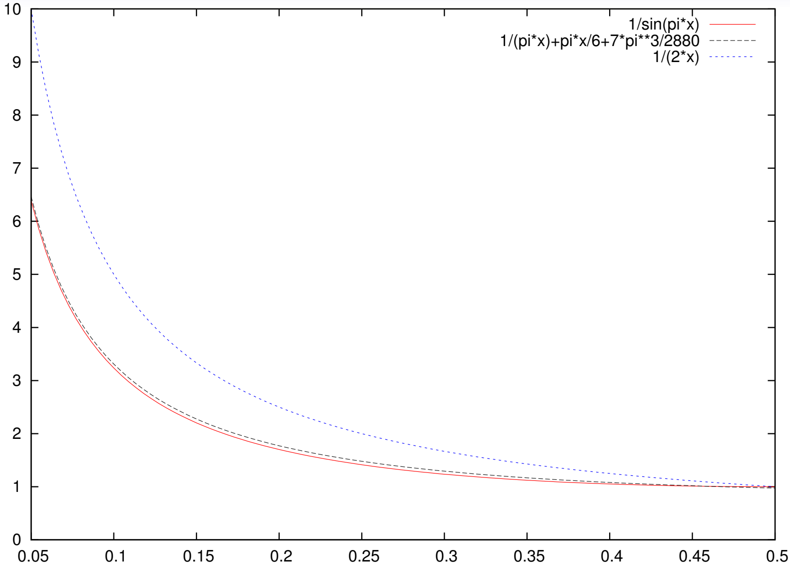
Since $x \in (0, 1/2]$, let us try

$$G(x) = \frac{1}{\pi x} + \frac{\pi x}{6} + \frac{7\pi^3}{2880}.$$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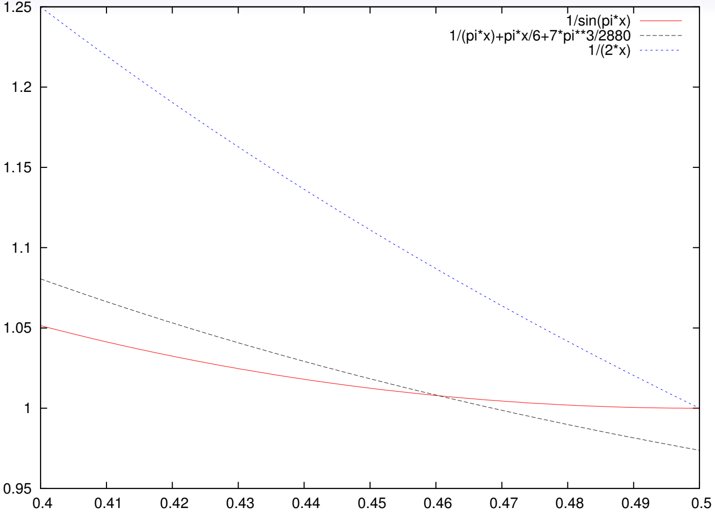

Numerically, red line goes above black line at $x = 0.4604264347$.

For our purposes, we shall use the approximation $\kappa = 0.46$.

Then, from the graph, we have

$$\frac{1}{\sin(\pi x)} < \frac{1}{\pi x} + \frac{\pi x}{6} + \frac{7\pi^3}{2880}, \quad x \in (0, \kappa].$$

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Mathematical proof uses undergraduate calculus.

For $x \in (\kappa, 1/2]$, we set $G(x) = c_1 + c_2 x$ such that G is continuous at $x = \kappa$ and $G(1/2) = 1$. Then

$$c_1 \approx 1.102449 \quad \text{and} \quad c_2 \approx -0.204898.$$

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Undergraduate calculus then leads to:

Theorem

Let $\kappa = 0.46$ *and let*

$$G(x) = \begin{cases} 1/(\pi x) + \pi x/6 + 7\pi^3/2880 & \text{for } x \in (0, \kappa], \\ c_1 + c_2 x & \text{for } x \in (\kappa, 1/2]. \end{cases}$$

Then G *satisfies* $1/\sin(\pi x) \le G(x) \le 1/(2x)$ *for* $x \in (0, 1/2]$.

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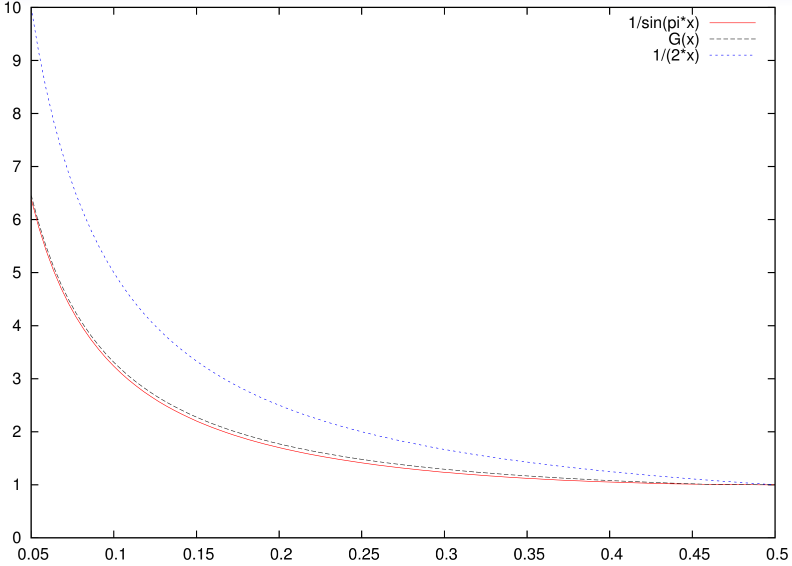
Remark: $1/(\pi x) + \pi x/6 + 1 - 2/\pi - \pi/12$ could be used for all $x \in (0, 1/2]$, but is not as close to $1/\sin(\pi x)$.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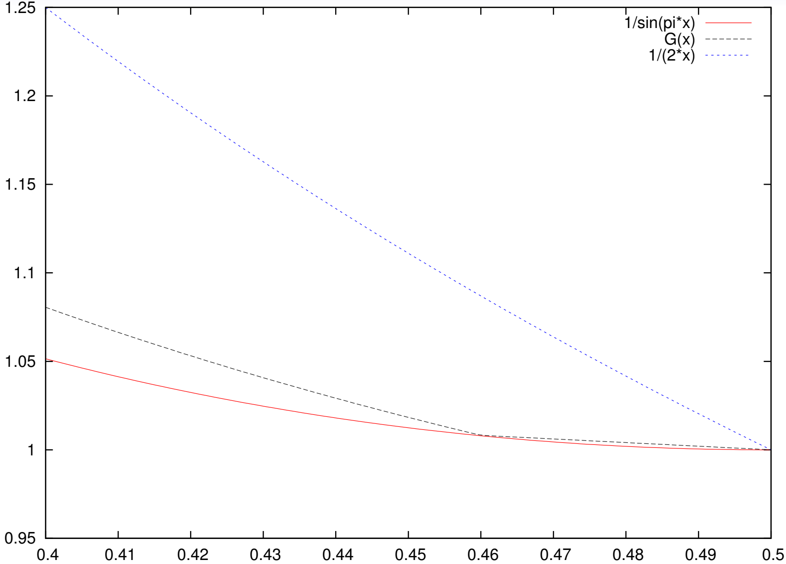
1/sin(pi*x)
G(x)
1/(2*x)
1.25
1.2
1.15
1.1
1.05
1
0.95
0.4
0.41
0.42
0.43
0.44
0.45
0.46
0.47
0.48
0.49
0.5

Numerical results

For $d = 2$ and $d = 3$, it is feasible to compute the star discrepancy.

Let $E(\boldsymbol{z}, n) = D^*(P_n(\boldsymbol{z})) - [1 - (1 - 1/n)^d]$. Then

$$E(\boldsymbol{z}, n) \le T(\boldsymbol{z}, n) \le W(\boldsymbol{z}, n) \le R(\boldsymbol{z}, n)/2.$$

Results for $d = 2$ using Fibonacci lattices.

n	$\boldsymbol{z}$	$E(\boldsymbol{z}, n)$	$T(\boldsymbol{z}, n)$	$W(\boldsymbol{z}, n)$	$R(\boldsymbol{z}, n)/2$
610	$(1, 377)$	$2.24(-3)$	$1.93(-2)$	$2.00(-2)$	$8.44(-2)$
987	$(1, 610)$	$1.51(-3)$	$1.35(-2)$	$1.40(-2)$	$5.95(-2)$
1597	$(1, 987)$	$1.01(-3)$	$9.39(-3)$	$9.70(-3)$	$4.16(-2)$
2584	$(1, 1597)$	$6.47(-4)$	$6.47(-3)$	$6.68(-3)$	$2.88(-2)$
4181	$(1, 2584)$	$4.70(-4)$	$4.44(-3)$	$4.58(-3)$	$1.99(-2)$
6765	$(1, 4181)$	$3.09(-4)$	$3.03(-3)$	$3.12(-3)$	$1.36(-2)$
10946	$(1, 6765)$	$2.02(-4)$	$2.06(-3)$	$2.11(-3)$	$9.29(-3)$

Results for $d = 3$ using $\boldsymbol{z} = (1, \ell, \ell^2 \mod n)$. The ℓ taken from Haber (1983).

n	ℓ	$E(\boldsymbol{z}, n)$	$T(\boldsymbol{z}, n)$	$W(\boldsymbol{z}, n)$	$R(\boldsymbol{z}, n)/2$
64	5	6.59(−2)	6.36(−1)	6.69(−1)	5.44(0)
96	39	5.84(−2)	6.50(−1)	6.80(−1)	4.55(0)
128	41	1.68(−2)	4.41(−1)	4.62(−1)	4.10(0)
192	39	4.09(−2)	4.45(−1)	4.64(−1)	3.36(0)
256	37	2.63(−2)	3.06(−1)	3.19(−1)	2.95(0)
384	81	2.17(−2)	2.86(−1)	2.98(−1)	2.36(0)
512	123	9.35(−3)	1.97(−1)	2.04(−1)	2.05(0)

Results for $d = 10$ using extensible $\boldsymbol{z}$ from Kuo website.

n	$T(\boldsymbol{z}, n)$	$W(\boldsymbol{z}, n)$	$R(\boldsymbol{z}, n)/2$
1024	2.44(4)	2.69(4)	2.19(8)
2048	2.64(4)	2.89(4)	2.71(8)
4096	2.71(4)	2.94(4)	3.11(8)
8192	2.64(4)	2.86(4)	3.35(8)
16384	2.48(4)	2.66(4)	3.41(8)

Results for $d = 20$ using extensible $\boldsymbol{z}$ from Kuo website.

n	$T(\boldsymbol{z}, n)$	$W(\boldsymbol{z}, n)$	$R(\boldsymbol{z}, n)/2$
1024	6.10(11)	7.41(11)	9.87(19)
2048	1.43(12)	1.71(12)	3.02(20)
4096	3.00(12)	3.55(12)	7.93(20)
8192	5.73(12)	6.70(12)	1.84(21)