Interpolation lattices for hyperbolic cross trigonometric polynomials

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Road map

Introduction

- $\mathbb{T} \simeq [0,1)$, $f \in C(\mathbb{T})$, $n \in \mathbb{N}$, $x_j = j/2^n$

$$\begin{aligned} \hat{f}_k &= \int_{\mathbb{T}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}x \\ &\approx \frac{1}{2^n} \sum_{j=0}^{2^n-1} f(x_j)\, \mathrm{e}^{-2\pi \mathrm{i} k x_j}, \quad k = -2^{n-1}+1, \ldots, 2^{n-1} \end{aligned}$$

- discrete Fourier transform (DFT)

$$f(x_j) = \sum_{k=-2^{n-1}+1}^{2^{n-1}} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} k x_j}, \quad j = 0, \ldots, 2^n - 1$$

- unitary up to a scaling factor
- complexity
  - DFT: $\mathcal{O}\left(2^{2n}\right)$
  - FFT: $\mathcal{O}\left(2^n n\right)$

Introduction

- discrete Fourier transform (d-dimensional)

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{k}\in\hat{G}_n^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k}\mathbf{x}} \\ \hat{G}_n^d &= \left\{-2^{n-1}+1,\ldots,2^{n-1}\right\}^d \subset \mathbb{Z}^d \\ \mathbf{x} &= \left(\frac{j_1}{2^n},\ldots,\frac{j_d}{2^n}\right)^T \in \mathbb{T}^d, \quad j_1,\ldots,j_d \in \{0,\ldots,2^n-1\} \end{aligned}$$

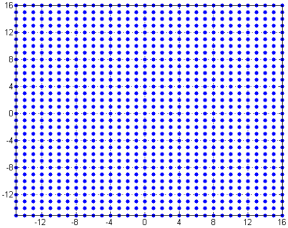
- unitary up to a scaling factor
- problem size $|\hat{G}_n^d| = 2^{nd}$, complexity
  - DFT: $\mathcal{O}\left(2^{2nd}\right)$ or $\mathcal{O}\left(2^{n(d+1)}\right)$
  - FFT: $\mathcal{O}\left(2^{nd} nd\right)$
- problem size and complexity increase strongly with dimension d

Introduction

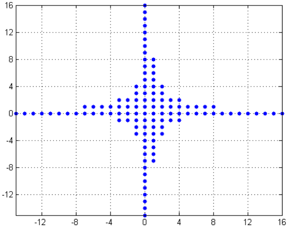
- evaluate trigonometric polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k}\mathbf{x}}$$

- aim:
  - stable spatial discretisation
  - fast algorithm



(a) full frequency grid $\hat{G}_5^2$



(b) hyperbolic cross H_5^2

Hyperbolic cross DFT (HCDFT)

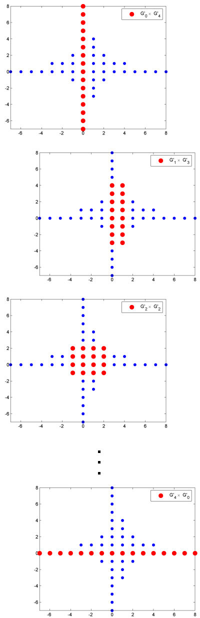
- one dimensional frequency grid

$$\hat{G}_n = \{-2^{n-1}+1, \ldots, 2^{n-1}\}, \quad \hat{G}_0 = \{0\}$$

- hyperbolic cross, dimension $d \in \mathbb{N}$, refinement $n \in \mathbb{N}_0$

$$H_n^d = \bigcup_{\substack{\mathbf{q} \in \mathbb{N}_0^d \\ \|\mathbf{q}\|_1 = n}} \hat{G}_{q_1} \times \ldots \times \hat{G}_{q_d}$$

- $\mathbf{k} \in H_n^d \ \Rightarrow \ |k_1 \cdots k_d| \leq 2^{n-d}$



Hyperbolic cross DFT (HCDFT)

- evaluation of

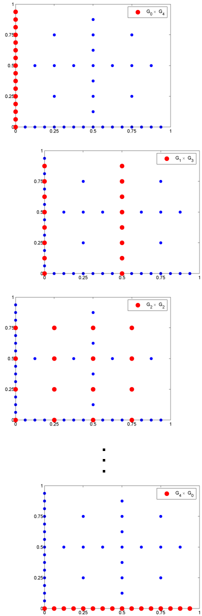
$$f(\mathbf{x}) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k}\mathbf{x}}, \qquad \mathbf{x} \in S_n^d$$

- one dimensional spatial grid

$$G_n = \left\{ \frac{j}{2^n} : j = 0, \ldots, 2^n - 1 \right\}$$

- sparse grid

$$S_n^d = \bigcup_{\substack{\mathbf{q} \in \mathbb{N}_0^d \\ \|\mathbf{q}\|_1 = n}} G_{q_1} \times \ldots \times G_{q_d}$$



Hyperbolic cross DFT (HCDFT)

- discrete Fourier transform on the hyperbolic cross

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}}, \qquad \mathbf{x} \in S_n^d$$

- matrix vector notation

$$\mathbf{F}_n^d := \left(\mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\mathbf{x}} \right)_{\mathbf{x} \in S_n^d, \mathbf{k} \in H_n^d}, \qquad \mathbf{f} = \mathbf{F}_n^d \hat{\mathbf{f}}$$

- problem size for fixed dimension d

$$\left| H_n^d \right| = \left| S_n^d \right| \;=\; \mathcal{O}\left(2^n n^{d-1} \right) \qquad (n \to \infty)$$

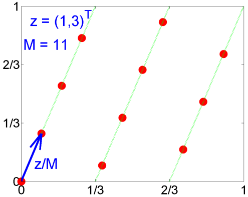
- complexity
  - HCDFT: $\mathcal{O}\left(2^{2n} n^{2d-2}\right)$ - naive algorithm
  - HCFFT: $\mathcal{O}\left(2^n n^d\right)$ - fast algorithm (Baszenski, Delvos 1989; Hallatschek 1992)
- increasing condition number (K., Kunis 2009)
  - fixed dimension d: $\mathrm{cond}(\mathbf{F}_n^d) \sim \sqrt{|H_n^d|}$
  - fixed refinement n: $\mathrm{cond}(\mathbf{F}_n^d) \sim |H_n^d|^2$

Lattice based HCFFT (LHCFFT)

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d$, $M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod \mathbf{1};\ j = 0, \ldots, M-1$$

- reformulation as 1-d DFT, $\mathbf{f} = \mathbf{A}\hat{\mathbf{f}}$



$$f(\mathbf{x}_j) = \sum_{\mathbf{k} \in H_n^d} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k}\mathbf{x}_j} = \sum_{l=0}^{M-1} \left(\sum_{\mathbf{k}\mathbf{z} \equiv l \mod M} \hat{f}_{\mathbf{k}} \right) e^{2\pi i \frac{j\mathbf{k}\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l e^{2\pi i \frac{jl}{M}}$$

- complexity $\mathcal{O}(M \log M + |H_n^d|)$

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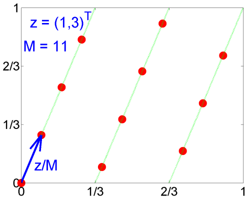

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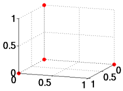
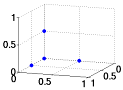
Aim: stable and unique reconstruction of $\hat{f}_{\mathbf{k}}$

LHCFFT - example fixed refinement $n = 1$

- HCDFT total refinement $n = 1$
  - $H_1^d = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$
  - $S_1^d = \frac{1}{2} H_1^d$, $|H_1^d| = |S_1^d| = d + 1$
  - $\mathbf{f} = \mathbf{F}_1^d \hat{\mathbf{f}}$, complexity $\mathcal{O}(d)$
  - $\frac{(d-2)^2}{2} \leq \mathrm{cond}(\mathbf{F}_1^d) \leq \frac{d^2}{2}$



H_1^3



S_1^3

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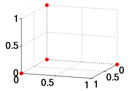
  - $\mathbf{f} = \mathbf{F}_1^d \hat{\mathbf{f}}$, complexity $\mathcal{O}(d)$
  - $\frac{(d-2)^2}{2} \leq \mathrm{cond}(\mathbf{F}_1^d) \leq \frac{d^2}{2}$
- a stable lattice $\tilde{S}_1^d$, $\mathbf{z} = (1, 2, \dots, d)^\top$, $M = d + 1$

$$\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod \mathbf{1}, \quad j = 0, \dots, d$$

- hyperbolic cross discrete Fourier transform

$$f_j = f(\mathbf{x}_j) = \sum_{\mathbf{k} \in H_1^d} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \mathbf{x}_j} = \sum_{l=0}^{d} \hat{f}_{\mathbf{k}_l} \mathrm{e}^{2\pi \mathrm{i} \frac{jl}{d+1}}, \qquad j = 0, \dots, d$$

- unitary up to a scaling factor, complexity $\mathcal{O}(d \log d)$



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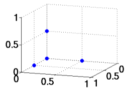
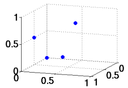


S_1^3

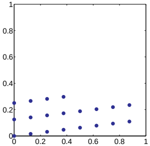


$\tilde{S}_1^3$

LHCFFT - example fixed dimension $d = 2$

- generating vector $\mathbf{z} \in \mathbb{R}^2$

$$\mathbf{z} = M(2^{-n}, 2^{-2n})^\top, M = |H_n^2|$$

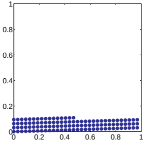


n=3

- corresponding $1d$ - Non-uniform DFT, $\mathbf{x}_j = \frac{j\mathbf{z}}{M} \mod 1$

$$f(\mathbf{x}_j) = \sum_{\mathbf{k} \in H_n^2} \hat{f}_{\mathbf{k}} e^{2\pi i \mathbf{k} \mathbf{x}_j} = \sum_{\mathbf{k} \in H_n^2} \hat{f}_{\mathbf{k}} e^{2\pi i j \frac{\mathbf{k}\mathbf{z}}{M}} = \sum_{y \in Y(H_n^d, \mathbf{z})} \hat{g}_y e^{2\pi i j y}$$

- $y = \sum_{s=1}^{2} 2^{-sn} k_s$ distinct
- unique reconstruction of Fourier coefficients $\hat{f}_{\mathbf{k}}$
- strongly growing condition number



n=5

Lattice based HCFFT (LHCFFT)

Theorem

Let $n, d \in \mathbb{N}$, a sampling scheme $\mathcal{X} = \{\mathbf{x}_j, j = 0, \ldots, M-1\}$, and let the Fourier matrix $\mathbf{A} = \left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\mathbf{x}_j}\right)_{j=0,\ldots,M-1;\mathbf{k}\in H_n^d}$ fulfil

$$\mathbf{A}^*\mathbf{A} = M\mathbf{I}.$$

1. *$\mathcal{X}$ is arbitrary , then $M \geq 2^{2n-2}$.*
2. *$\mathcal{X}$ is a rank-1 lattice, then $\mathbf{z} \in \mathbb{Z}^d$ and* $\mathbf{k}_1\mathbf{z} \not\equiv \mathbf{k}_2\mathbf{z} \mod M$ *for all $\mathbf{k}_1, \mathbf{k_2} \in H_n^d$, $\mathbf{k}_1 \neq \mathbf{k}_2$.*

- complexity of stable and nonaliasing LHCFFT larger than $c2^{2n-2}$

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- complexity of stable and nonaliasing LHCFFT larger than $c2^{2n-2}$
- for comparison:
  - $|H_n^d| = \mathcal{O}(2^n n^{d-1})$
  - complexity of sparse grid HCFFT $\mathcal{O}(2^n n^d)$

LHCFFT - interpolation & integration

- Zaremba cross
$$\tilde{H}_n^d = \{\mathbf{k} \in \mathbb{Z}^d : \prod_{s=1}^{d} \max(1, |k_s|) \leq 2^n\}$$
- $\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\mathbf{x}} d\mathbf{x} = \int_{\mathbb{T}} \sum_{\mathbf{h} \in H_n^d} \hat{f}_{\mathbf{h}} e^{2\pi i (\mathbf{h}-\mathbf{k})\mathbf{x}} d\mathbf{x}, \ \mathbf{k} \in H_n^d$
- $\{\mathbf{l} \in \mathbb{Z}^d : \mathbf{l} = \mathbf{h} - \mathbf{k} : \mathbf{h}, \mathbf{k} \in H_n^d\} \subsetneq \tilde{H}_{2n-2}^d$

M	Zaremba index	n	Reference
4002	280	5	Lyness, Sørevik 1993
3052	110	6	K., Kunis, Potts 2010
6066	460	6	Lyness, Sørevik 1993

Interpolation lattice sizes M for H_n^3

LHCFFT - generating vectors of Korobov form

- generating vectors of Korobov form, $a \in \mathbb{N}$

$$\mathbf{z}(a) = (1, a, a^2, \ldots, a^{d-1})^\top \in \mathbb{N}^d$$

Lemma

Let $d, n, a \in \mathbb{N}$ be given, and $\mathbf{z}(a)$ of Korobov form, then the following holds true

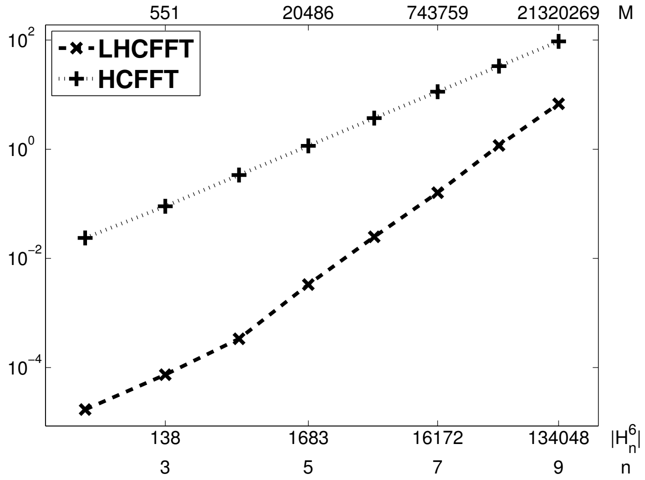
1. *if and only if $3 \cdot 2^{n-2} \leq a$, $(\mathbf{kz})_{\mathbf{k} \in H_n^d}$ contains distinct values,*
2. *if $d = 2$, $a = 3 \cdot 2^{n-2}$ then $M^* = (1 + 3 \cdot 2^{n-2})2^{n-1}$ is the minimal M with $(\mathbf{kz} \mod M)_{\mathbf{k} \in H_n^d}$ is distinct.*

LHCFFT - fixed spatial dimension $d = 2$

n	$\lvert H_n^2 \rvert$	M^*	α	HCFFT	LHCFFT
2	8	8	1.0	0.005	0.000
3	20	28	1.4	0.005	0.000
4	48	104	2.2	0.011	0.000
5	112	400	3.6	0.025	0.000
6	256	1568	6.1	0.055	0.000
8	1280	24704	19.3	0.258	0.005
10	6144	393728	64.1	1.218	0.082
12	28672	6293504	219.5	5.582	1.470
14	131072	100671488	768.1	24.232	33.114

minimal lattice sizes for $\mathbf{z}(3 \cdot 2^{n-2}) = (1, 3 \cdot 2^{n-2})^\top$

LHCFFT - fixed spatial dimension $d = 6$



Computational times of HCFFT and LHCFFT, $\mathbf{z}(3 \cdot 2^{n-2})$

Summary

- full grid FFT - suffers from 'curse of dimensionality'
- HCFFT decreases problem size and complexity strongly
- condition number increases strongly for standard sparse grids
- rank-1 lattices
  - stable spatial discretisation
  - expensive search for small lattices
  - growing oversampling
  - fast for reasonable problem sizes

NHCFFT Matlab Toolbox http://www.tu-chemnitz.de/∼lkae