Small deviations of smooth Gaussian processes

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joint work with Frank Aurzada (Berlin), Fuchang Gao (Moscow, Idaho), Wenbo Li (Newark, Delaware) and Qiman Shao (Hong Kong)

Main aim of the talk

- congratulate Stefan to his birthday

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- determine the exact asymptotic behaviour of small deviations of certain smooth Gaussian processes
- illustrate the connection between small deviations and metric entropy

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Plan of the talk

1. The small deviation problem for stochastic processes
2. Metric entropy
3. Relations between small deviations and metric entropy
4. Small deviations under L_2-norm
5. Small deviations under $\sup -$norm

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- General Problem.
  Let $X = X(t)\,, 0 \le t \le 1\,$, be a random process attaining values in some Banach space E, in most cases $E = L_2[0,1]$ or $C[0,1]$.
  For $\varepsilon > 0$ consider the small deviation probabilities on the $\log$ −level,

$$\phi(\varepsilon) = \phi_X(\varepsilon) := -\log \mathbb{P}\left(\|X\|_E \le \varepsilon\right)$$

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- Applications in probability and analysis.
  - law of the iterated logarithm of Chung type
  - strong limit laws in statistics
  - quantization (approximation) of stochastic processes
  - metric entropy of linear operators

- Consider the family of centered Gaussian processes $X_{\alpha,\beta}(t)\,, t \geq 0\,,$ defined by the covariance function

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- Special problem.
  (posed at the Workshop in Palo Alto, December 2008)
  Find the small deviation rates of the processes $X_{\alpha,\beta}$ w.r.t. the
  – L_2 norm, if $\alpha > 0$ and $-1/2 < \beta < \alpha$
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- The main tool for answering this question is the close relation to metric entropy.

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- For a subset K of a metric space (E,d) we define its entropy numbers and covering numbers

$$\begin{aligned} \varepsilon_n(K) = &\quad \inf\{\varepsilon > 0 : K \text{ can be covered by } n \text{ balls of radius } \varepsilon\} \\ N(K,\varepsilon) = &\quad \min\{n \in \mathbb{N} : K \text{ can be covered by } n \text{ balls of radius } \varepsilon\} \end{aligned}$$

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- The (dyadic) entropy numbers of an operator $T : X \to Y$ between Banach spaces are defined as
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- Remark. A set K is relatively compact $\iff \lim\limits_{n\to\infty} \varepsilon_n(K) = 0$
  An operator T is compact $\iff \lim\limits_{n\to\infty} e_n(T) = 0.$
  $\curvearrowright$ The asymptotic behaviour of these quantities is a measure for the "degree" of compactness of sets/operators.

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- μ is uniquely determined by each of the following quantities:
  – characteristic function $\widehat{\mu}(a) := \int_E e^{ia(x)}\, d\mu(x)\,, a \in E'$
  – covariance operator $R : E' \to E$, $Ra := \underbrace{\int_E xa(x)\, d\mu(x)}_{\text{Bochner integral}}$
  – RKHS H_μ, i.e. the completion of $R(E')$ w.r.t. the inner product

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  $$\langle Ra, Rb\rangle_\mu = \int_E a(x)b(x)\, d\mu(x)\,, \quad a, b \in E'$$

- $\curvearrowright$ $\widehat{\mu}(a) = \exp(-\frac{1}{2}\langle Ra, Ra\rangle_\mu) = \exp(-\frac{1}{2}\|S'a\|_H^2)$
  for some operator $S : H \to E$, where H is a Hilbert space.
  $\curvearrowright$ $R = SS'$, and $S(B_H)$ is dense in the unit ball K_μ of H_μ.
  Consequently, both sets have the same $\varepsilon-$entropy

- Kuelbs-Li 1993. Let $\phi(\varepsilon) = \phi_\mu(\varepsilon) := -\log \mu(\{x \in E : \|x\| \le \varepsilon\})$. Then

$$(i) \quad H(2\varepsilon/\lambda, K_\mu) \le \phi(\varepsilon) + \lambda^2/2$$

and $$(ii) \quad H(\varepsilon/\lambda, K_\mu) \ge \phi(2\varepsilon) + \log \Phi(\lambda + \alpha_\varepsilon) \qquad \text{for all } \lambda, \varepsilon > 0\,,$$

where $\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2} dt$

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- Some consequences. ($\sim$ means equivalence up to constants)
  1. Goodman 1990. $H(\varepsilon, K_\mu) = o(\varepsilon^{-2}) \quad \curvearrowright K_\mu$ is compact in E.
  2. Li-Linde 1999. $\phi(\varepsilon) \sim \varepsilon^{-\alpha} \iff H(\varepsilon, K_\mu) \sim \varepsilon^{-2\alpha/(2+\alpha)}$
  3. Aurzada-Ibragimov-Lifshits-van Zanten 2008.

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$$\phi(\varepsilon) \sim (\log 1/\varepsilon)^\alpha \Longleftrightarrow H(\varepsilon, K_\mu) \sim (\log 1/\varepsilon)^\alpha$$

- Translation to processes. Let X be a Gaussian process that attains values in E and is related to an operator $S : H \to E$ via

$$(*) \qquad \mathbb{E}e^{ia(X)} = \exp(-\tfrac{1}{2}\|S'a\|^2)\,, a \in E'\,.$$

$\curvearrowright$ For the Gaussian measure μ on E generated by X we have

$$\underbrace{\phi_\mu(\varepsilon)}_{=\phi_X(\varepsilon)} \sim (\log 1/\varepsilon)^\alpha \iff \underbrace{H(\varepsilon, K_\mu)}_{=H(\varepsilon, S(B_H))} \sim (\log 1/\varepsilon)^\alpha$$

$$\iff -\log e_n(S : H \to E) \sim n^{-1/\alpha}$$

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- It is easy to check that the processes $X_{\alpha,\beta} = X_{\alpha,\beta}(t), 0 \leq t \leq 1$ is related – via formula (*) – to the operator

(**) $$(Sf)(t) = t^\alpha \int_0^\infty x^\beta e^{-xt} f(x)\, dx\,, \quad f \in L_2[0,\infty), t \in [0,1]$$

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4. Small deviations under L_2-norm

- Theorem 1. Let $\alpha > 0$ and $-1/2 < \beta < \alpha$. Then

$$-\log \mathbb{P}\left(\int_0^1 X_{\alpha,\beta}(t)^2\,dt \le \varepsilon^2\right) \sim (\log 1/\varepsilon)^3 \quad \text{as } \varepsilon \to 0\,.$$

From our previous considerations it is clear that Theorem 1 is a consequence of the following

Proposition 2. For $\alpha > \beta > -1/2$, the entropy numbers of the operator $S : L_2[0,\infty) \to L_2[0,1]$ given by (**) satisfy

$$-\log e_n(S) \sim \sqrt[3]{n} \quad \text{as } n \to \infty\,.$$

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Sketch of proof. The operator $T := SS' : L_2[0,1] \to L_2[0,1]$ is given by

$$(Tf)(t) = \Gamma(2\beta+1) \int_0^1 \frac{(tx)^\alpha}{(t+x)^{2\beta+1}}\, dt\,.$$

The singular numbers of T are known (Laptev 1974),

$$-\log s_n(T) \approx 2\pi\sqrt{(\alpha-\beta)n} \quad \text{(strong equivalence } \approx\text{)}\,,$$

and from $s_n(T) = s_n(S)^2$ we get, with $c = \pi\sqrt{\alpha-\beta}$,

$$-\log s_n(S) \approx c\sqrt{n}\,.$$

If D_σ is a diagonal operator in ℓ_2 with diagonal entries $\sigma_n \searrow 0$, then (a special case of results in Gordon-König Schütt 1987),

$$\sup_{n\geq 1}\left(2^{-k/n}(\sigma_1\cdots\sigma_n)^{1/n}\right) \leq e_{k+1}(D_\sigma) \leq 6\sup_{n\geq 1}\left(2^{-k/n}(\sigma_1\cdots\sigma_n)^{1/n}\right) .$$

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In our case:

$$-\log e_k(S) \approx \sup_{n\geq 1}\left(\frac{k\log 2}{n} + \frac{c}{n}\sum_{j=1}^{n}\sqrt{j}\right)$$

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Finally we optimize over n. Taking $n \sim k^{2/3}$ we get

$$-\log e_k(S) \sim k^{1/3}$$

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$$-\log e_k(S) \sim k^{1/3} \qquad \square$$

Remark. The proof gives even the exact small deviation constant ,

$$\lim_{\varepsilon\to 0}\frac{-\log \mathbb{P}(\|X_{\alpha,\beta}\|_2 \leq \varepsilon)}{(\log 1/\varepsilon)^3} = \frac{1}{3(\alpha-\beta)\pi^2}.$$

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5. Small deviations under sup-norm

- Theorem 3. Let $\alpha > \beta + 1/2 > 0$. Then

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- Using again the "entropy connection", it is enough to show

  Proposition 4. Let $\alpha > \beta + 1/2 > 0$, and let the operator $S : L_2[0,\infty) \to C[0,1]$ be given by

$$(Sf)(t) = t^\alpha \int_0^\infty x^\beta e^{-xt} f(x)\,dx\,.$$

  Then

$$-\log e_n(S : L_2[0,\infty) \to C[0,1]) \sim n^{1/3}\,.$$

The proof of the upper bound is simple. Obviously

$$e_n(S : L_2[0,\infty) \to C[0,1]) \geq e_n(S : L_2[0,\infty) \to L_2[0,1]),$$

and Proposition 2 gives immediately

$$-\log e_n(S : L_2 \to C[0,1]) \leq -\log e_n(S : L \to L_2[0,1]) \sim n^{1/3}.$$

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The proof of the lower bound is less trivial. Starting point is the observation that S maps $L_2[0,\infty)$ even into a smaller space than $C[0,1]$, namely the Hölder space $C^\lambda[0,1]$ where $\lambda = \min(\alpha - \beta - 1/2, 1/2)$. It consists of all functions $f \in C[0,1]$ with finite norm

$$\|f\|_{C^\lambda[0,1]} := \sup_{0\leq s<t\leq 1} \frac{|f(t)-f(s)|}{|t-s|^\lambda} + \sup_{0\leq t\leq 1} |f(t)|\,.$$

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$$\|f\|_{C^\lambda[0,1]} := \sup_{0\leq s<t\leq 1} \frac{|f(t)-f(s)|}{|t-s|^\lambda} + \sup_{0\leq t\leq 1} |f(t)|\,.$$

But how can we take advantage of this fact? The answer is given by the following result, which might be of independent interest.

Lemma . Let $T : E \to C^\lambda[0,1]$ be a bounded linear operator, where E is a Banach space and $0 < \lambda \leq 1$. Then, for some $c > 0$ and all $n \in \mathbb{N}$,

$$e_n(T : E \to C[0,1]) \leq c e_n(T : E \to L_2[0,1])^{\lambda/(\lambda+1/2)}$$

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Ingredients of the proof. We use the smoothing/averaging operators $P_\delta : L_2[0,1] \to C[0,1]$,

$$P_\delta f(t) = \frac{1}{|I_\delta(t)|} \int_{I_\delta(t)} f(x)\, dx\,,$$

over "moving" intervals $I_\delta(t) = [t-\delta, t+\delta] \cap [0,1]\,,$

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for which we have the simple norm estimates

$$\|P_\delta : L_2[0,1] \to C[0,1]\| \le \delta^{-1/2}$$
$$\|id - P_\delta : C^\lambda[0,1] \to C[0,1]\| \le \delta^\lambda$$

By elementary properties of entropy numbers we obtain

$$\begin{aligned} e_n(T : E \to C) &\leq e_n(P_\delta T : E \to C) + \|T - P_\delta T : E \to C\| \\ &\leq e_n(T : E \to L_2)\|P_\delta : L_2 \to C\| \\ &+ \|T : E \to C^\lambda\|\|id - P_\delta : C^\lambda \to C\| \\ &\leq e_n(T : E \to L_2) \cdot \delta^{-1/2} + \|T : E \to C^\lambda\| \cdot \delta^\lambda \end{aligned}$$

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Finally, for large enough n we can optimize δ and obtain the result. $\square$

By elementary properties of entropy numbers we obtain

$$\begin{aligned} e_n(T : E \to C) &\le e_n(P_\delta T : E \to C) + \|T - P_\delta T : E \to C\| \\ &\le e_n(T : E \to L_2)\|P_\delta : L_2 \to C\| \\ &+ \|T : E \to C^\lambda\|\|id - P_\delta : C^\lambda \to C\| \\ &\le e_n(T : E \to L_2) \cdot \delta^{-1/2} + \|T : E \to C^\lambda\| \cdot \delta^\lambda \end{aligned}$$

Finally, for large enough n we can optimize δ and obtain the result. □

Open question.

$$\lim_{\varepsilon \to 0} \frac{-\log \mathbb{P}\left(\sup_{0 \le t \le 1} |X_{\alpha,\beta}(t)| \le \varepsilon\right)}{(\log 1/\varepsilon)^3} = ??$$

THANK YOU FOR YOUR ATTENTION!