Randomized Quasi-Monte Carlo: Theory, Choice of Discrepancy, and Applications

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Randomized Quasi-Monte Carlo: Theory, Choice of Discrepancy, and Applications featuring randomly-shifted lattice rules

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1. Monte Carlo (MC), quasi-MC (QMC), randomized QMC (RQMC).
2. Lattice rules and RQMC variance.
3. Weighted discrepancies and choice of weights.
4. Several examples.
Basic Monte Carlo setting

Want to estimate

$$\mu = \mu(f) = \int_{[0,1)^s} f(u) \, du = \mathbb{E}[f(U)]$$

where $f : [0, 1)^s \to \mathbb{R}$ and $U$ is a uniform r.v. over $[0, 1)^s$.

Standard Monte Carlo:

- Generate $n$ independent copies of $U$, say $U_1, \ldots, U_n$;
- estimate $\mu$ by $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} f(U_i)$. 
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Almost sure convergence as $n \to \infty$ (strong law of large numbers).

For confidence interval of level $1 - \alpha$, can use central limit theorem:

$$\mathbb{P} \left[ \mu \in \left( \frac{\hat{\mu}_n - c_\alpha S_n}{\sqrt{n}}, \frac{\hat{\mu}_n + c_\alpha S_n}{\sqrt{n}} \right) \right] \approx 1 - \alpha,$$

where $S_n^2$ is any consistent estimator of $\sigma^2 = \text{Var}[f(U)]$. 
Quasi-Monte Carlo (QMC)

Replace the random points $U_i$ by a set of deterministic points $P_n = \{u_0, \ldots, u_{n-1}\}$ that cover $[0, 1)^s$ more evenly. This $P_n$ is called a highly-uniform or low-discrepancy point set if some measure of discrepancy between the empirical distribution of $P_n$ and the uniform distribution $\to 0$ faster than for independent random points.
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**Main construction methods:** lattice rules and digital nets (Korobov, Hammersley, Halton, Sobol’, Faure, Niederreiter, etc.)
Simplistic solution: rectangular grid

\[ P_n = \{(i_1/d, \ldots, i_s/d) \text{ such that } 0 \leq i_j < d \ \forall j\} \text{ where } n = d^s. \]
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Quickly becomes impractical when \( s \) increases.
And each one-dimensional projection has only \( d \) distinct points, each
two-dimensional projections has only \( d^2 \) distinct points, etc.
Example: lattice with $s = 2$, $n = 101$, $a = 12$

\[ P_n = \{(x/m, (ax/m) \mod 1) : x = 0, \ldots, m - 1\} \]
\[ = \{(x/101, (12x/101) \mod 1) : x = 0, \ldots, 100\} \]

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Two problems: (1) point at \((0, 0)\) and (2) how to estimate the error?
Randomized quasi-Monte Carlo (RQMC)

An RQMC estimator of $\mu$ has the form

$$\hat{\mu}_{n,\text{rqmc}} = \frac{1}{n} \sum_{i=0}^{n-1} f(U_i),$$

with $P_n = \{U_0, \ldots, U_{n-1}\} \subset (0, 1)^s$ an RQMC point set:

(i) each point $U_i$ has the uniform distribution over $(0, 1)^s$;
(ii) $P_n$ as a whole is a low-discrepancy point set.

$$\mathbb{E}[\hat{\mu}_{n,\text{rqmc}}] = \mu \quad \text{(unbiased)}.$$

Can perform $m$ independent realizations $X_1, \ldots, X_m$ of $\hat{\mu}_{n,\text{rqmc}}$, then estimate $\mu$ and $\text{Var}[\hat{\mu}_{n,\text{rqmc}}]$ by their sample mean $\bar{X}_m$ and sample variance $S_m^2$ (also unbiased).

Temptation: assume that $\bar{X}_m$ has the normal distribution.
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Generalized antithetic variates and RQMC

\[
\text{Var}[\hat{\mu}_{n,\text{rqmc}}] = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \text{Cov}[f(U_i), f(U_j)]
\]

\[
= \frac{\text{Var}[f(U_i)]}{n} + \frac{2}{n^2} \sum_{i<j} \text{Cov}[f(U_i), f(U_j)].
\]

We want to make the last sum as negative as possible.

Special cases:
- antithetic variates \((n = 2)\),
- Latin hypercube sampling (LHS),
- randomized quasi-Monte Carlo (RQMC).
Lattice rules

Integration lattice:

\[ L_s = \left\{ v = \sum_{j=1}^{s} z_j v_j \text{ such that each } z_j \in \mathbb{Z} \right\}, \]

where \( v_1, \ldots, v_s \in \mathbb{R}^s \) are linearly independent over \( \mathbb{R} \) and where \( L_s \) contains \( \mathbb{Z}^s \). Lattice rule: Take \( P_n = \{u_0, \ldots, u_{n-1}\} = L_s \cap [0,1)^s \).
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Lattice rule of rank 1: \( u_i = i v_1 \mod 1 \) for \( i = 0, \ldots, n - 1 \), where \( n v_1 = z = (z_1, \ldots, z_s) \in \{0, 1, \ldots, n - 1\} \).

Korobov rule: \( z = (1, a, a^2 \mod n, \ldots) \).
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Random shift modulo 1: generate a single point \( \mathbf{U} \) uniformly over \( (0, 1)^s \) and add it to each point of \( P_n \), modulo 1, coordinate-wise: \( \mathbf{U}_i = (\mathbf{u}_i + \mathbf{U}) \mod 1 \). Each \( \mathbf{U}_i \) is uniformly distributed over \([0, 1)^s\).
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Perhaps less obvious: Can generate it in any of the colored shapes below.
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Proposition. Let $R \subset [0, 1)^s$ such that

$$\{ R_i = (R + u_i) \bmod 1, \ i = 0, \ldots, n-1 \}$$

is a partition of $[0, 1)^s$ in $n$ regions of volume $1/n$. Then, sampling the random shift $U$ uniformly in any given $R_i$ is equivalent to sampling it uniformly in $[0, 1)^s$.

The error function

$$g_n(U) = \hat{\mu}_{n, \text{rqmc}} - \mu$$

over any $R_i$ is the same as over $R$.  

Error function $g_n(u)$ for $f(u_1, u_2) = (u_1 - 1/2)(u_2 - 1/2)$. 

![Color map showing the error function $g_n(u)$ for the given function $f(u_1, u_2)$. The map ranges from -0.2 to 0.3 on the color scale, with the $u_1$ and $u_2$ axes labeled.]
Error function $g_n(u)$ for $f(u_1, u_2) = (u_1 - 1/2) + (u_2 - 1/2)$. 
Error function $g_n(u)$ for $f(u_1, u_2) = u_1 u_2 (u_1 - 1/2) (u_2 - 1/2)$. 
Variance expression

Suppose $f$ has Fourier expansion

$$ f(u) = \sum_{h \in \mathbb{Z}^s} \hat{f}(h) e^{2\pi \sqrt{-1} h^t u}. $$

For a randomly shifted lattice, the exact variance is (always)

$$ \text{Var}[\hat{\mu}_{n,rqmc}] = \sum_{0 \neq h \in L^*_s} |\hat{f}(h)|^2, $$

where $L^*_s = \{ h \in \mathbb{R}^s : h^t v \in \mathbb{Z} \text{ for all } v \in L_s \} \subseteq \mathbb{Z}^s$ is the dual lattice.

From the viewpoint of variance reduction, an optimal lattice for $f$ minimizes the square “discrepancy” $D^2(P_n) = \text{Var}[\hat{\mu}_{n,rqmc}].$
\[
\text{Var}[\hat{\mu}_{n, \text{rqmc}}] = \sum_{0 \neq h \in L_s^*} |\hat{f}(h)|^2.
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If \( f \) has square-integrable mixed partial derivatives up to order \( \alpha \), and the periodic continuation of its derivatives up to order \( \alpha - 1 \) is continuous across the unit cube boundaries, then

\[
|\hat{f}(h)|^2 = \mathcal{O}((\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}).
\]

Moreover, there is a \( v_1 = v_1(n) \) such that

\[
P_{2\alpha} \overset{\text{def}}{=} \sum_{0 \neq h \in L_s^*} (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha} = \mathcal{O}(n^{-2\alpha+\delta}).
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This is the variance for a worst-case \( f \) having

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Beware of hidden factor in \( \mathcal{O} \) when \( s \) is large. This worst-case function may be far from representative in applications.
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Baker’s transformation
Want to make the periodic continuation continuous.

If \( f(0) \neq f(1), \) define \( \tilde{f} \) by \( \tilde{f}(1 - u) = \tilde{f}(u) = f(2u) \) for \( 0 \leq u \leq 1/2. \) This \( \tilde{f} \) has the same integral as \( f \) and \( \tilde{f}(0) = \tilde{f}(1). \)
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For smooth \( f \), can reduce the variance to \( O(n^{-4+\delta}) \) (Hickernell 2002).
The resulting \( \tilde{f} \) also symmetric with respect to \( u = 1/2 \).
In practice, we transform the points \( U_i \) instead of \( f \).
One-dimensional case

Random shift followed by **baker’s transformation**. Along each coordinate, stretch everything by a factor of 2 and fold. Same as replacing $U_j$ by $\min[2U_j, 2(1 - U_j)]$.
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$U/n$
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![Diagram of points on a line]
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Along each coordinate, stretch everything by a factor of 2 and fold.
Same as replacing $U_j$ by $\min[2U_j, 2(1 - U_j)]$.
Gives locally antithetic points in intervals of size $2/n$. 

\begin{center}
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Searching for a lattice that minimizes

$$\text{Var}[\hat{\mu}_{n,\text{rqmc}}] = \sum_{0 \neq h \in L_s^*} |\hat{f}(h)|^2$$

is unpractical, because:

- the Fourier coefficients are usually unknown,
- there are infinitely many,
- must do it for each $f$.

We nevertheless want to see how far we can go in that direction.
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We start with a simple function for which we know the Fourier expansion. Even then, the discrepancy involves an infinite number of terms!

**Possible ideas:** Truncate the sum to a finite subset $B$:

$$\sum_{0 \neq h \in L_s^* \cap B} |\hat{f}(h)|^2,$$

or to the largest $q$ square coefficients $|\hat{f}(h)|^2$. But hard to implement!
Dual-space exploration

The following makes sense if the $|\hat{f}(h)|^2$ tend to decrease with each $|h_j|$. Start with a large set $\mathcal{L}$ of lattices (or generating vectors $v_1$, for given $n$). Search for vectors $h$ with large weights $w(h) = |\hat{f}(h)|^2$, via a neighborhood search starting at $h = 0$, keeping a sorted list (as in Dijkstra’s shortest path algorithm), and eliminate (successively) from $\mathcal{L}$ the lattices whose dual contains $h$ for the next largest $w(h)$, until a single lattice remains.

Example of neighborhood $\mathcal{N}(h)$: only one coordinate differs, by one unit.
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Component-by-component version: For $j = 1, 2, \ldots, s$, we apply the algorithm for a set $\mathcal{L}$ of $j$-dimensional lattices with common (fixed) $j - 1$ first coordinates, and determine the $j$th coordinate by visiting the $j$-dimensional vectors $h$.

When the $|\hat{f}(h)|$ are unknown, we may think of estimating them as needed, dynamically.
Algorithm Dual-Space-Exploration(lattice set \( \mathcal{L} \), weights \( w \));
\[
\begin{align*}
\mathcal{Q} & \leftarrow \mathcal{N}(0) \quad \text{// vectors } \mathbf{h} \text{ to be visited, sorted by weight } w(\mathbf{h}); \\
\mathcal{M} & \leftarrow \mathcal{N}(0) \quad \text{// vectors } \mathbf{h} \text{ who already entered } \mathcal{Q}; \\
\text{while } |\mathcal{L}| > 1 \text{ do} & \\
& \quad \mathbf{h} \leftarrow \text{remove first from } \mathcal{Q}; \\
& \quad \text{for all lattices } L_s \in \mathcal{L} \text{ such that } \mathbf{h} \in L_s^* \text{ do} \\
& \quad \quad \text{remove } L_s \text{ from } \mathcal{L}; \\
& \quad \quad \text{if } |\mathcal{L}| = 1 \text{ then} \\
& \quad \quad \quad \text{return the single lattice } L_s \in \mathcal{L} \text{ and exit;} \\
& \quad \quad \text{end if} \\
& \quad \text{end for} \\
& \quad \text{for all } \mathbf{h}' \in \mathcal{N}(\mathbf{h}) \setminus \mathcal{M} \text{ do} \\
& \quad \quad \text{add } \mathbf{h}' \text{ to } \mathcal{M} \text{ and to } \mathcal{Q} \text{ with priority (weight) } w(\mathbf{h}'); \\
& \quad \text{end for} \\
& \text{end while}
\end{align*}
\]
An example

Take the product V-shaped function

\[ f(u) = \prod_{j=1}^{s} \frac{|4u_j - 2| + c_j}{1 + c_j}, \]

so

\[ \hat{f}(h) = \prod_{\{j : h_j \text{ is odd}\}} \frac{4}{(1 + c_j)\pi^2 h_j^2}. \]

Dimensions \( s = 5 \) and 10.
Constants \( c_j = 1, j, j^2, j^3 \).
The Dual Exploration Algorithm in Action
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### The Dual Exploration Algorithm in Action

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The Dual Exploration Algorithm in Action
Evolution of the Dual Exploration Algorithm

\[ s = 2, \ c_j = j, \ n = 1021 \]
Estimated variance vs $n$ for $s = 5$

- $s = 5, c_j = 1$
- $s = 5, c_j = j$
- $s = 5, c_j = j^2$
- $s = 5, c_j = j^3$
Estimated variance vs $n$ for $s = 10$

- $s = 10$, $c_j = 1$
- $s = 10$, $c_j = j$
- $s = 10$, $c_j = j^2$
- $s = 10$, $c_j = j^3$
ANOVA decomposition

The Fourier expansion has too many terms to handle. As a cruder expansion, we can write $f(u) = f(u_1, \ldots, u_s)$ as:

$$f(u) = \sum_{u \subseteq \{1, \ldots, s\}} f_u(u) = \mu + \sum_{i=1}^{s} f_{\{i\}}(u_i) + \sum_{i,j=1}^{s} f_{\{i,j\}}(u_i, u_j) + \cdots$$

where

$$f_u(u) = \int_{[0,1]^{|\bar{u}|}} f(u) \, du - \sum_{v \subset u} f_v(u_v),$$

and the Monte Carlo variance decomposes as

$$\sigma^2 = \sum_{u \subseteq \{1, \ldots, s\}} \sigma^2_u, \quad \text{where } \sigma^2_u = \text{Var}[f_u(U)].$$

Sensitivity indices: $S_u = \sigma^2_u / \sigma^2$. Can be estimated by MC or RQMC.

Heuristic intuition: Make sure the projections of $P_n$ are very uniform for the important subsets $u$ (i.e., with large $S_u$).
Shift-invariant discrepancy

In a reproducing kernel Hilbert space (RKHS) with kernel $K$, and randomly-shifted points, the relevant discrepancy corresponds to the shift-invariant kernel

$$K_{sh}(u_i, u_j) := \mathbb{E}[K(U_i, U_j)] = \mathbb{E}[K(u_i + U, u_j + U)] = \mathbb{E}[K(u_i - u_j + U, U)].$$

The mean square discrepancy can be written as

$$\mathbb{E}[D^2(P_n)] = \frac{1}{n} \sum_{0 \neq h \in \mathbb{Z}^s} w(h) \sum_{i=0}^{n-1} e^{2\pi \sqrt{-1} h^t u_i} = \sum_{0 \neq h \in L_*^s} w(h)$$

(for a lattice).

Key issue: choice of the weights $w(h)$. 
Regrouping by projections: projection weights

Denote $u(h) = u(h_1, \ldots, h_s)$ the set of indices $j$ for which $h_j \neq 0$. We have

$$\mathbb{E}[D^2(P_n)] = \sum_{u \subseteq \{1, \ldots, s\}} \sum_{h : u(h) = u} w(h) \frac{1}{n} \sum_{i=0}^{n-1} e^{2\pi \sqrt{-1} h^t u_i} = \sum_{u \subseteq \{1, \ldots, s\}} D_u^2(P_n).$$

The RKHS decomposes as a direct sum and the RQMC variance has a corresponding decomposition.

$$\text{Var}[\hat{\mu}_{n, \text{rqmc}}] = \sum_{u \subseteq \{1, \ldots, s\}} \text{Var}[\hat{\mu}_{n, \text{rqmc}}(f_u)].$$

Restriction on the weights: take $w(h) = \gamma_u(h) \prod_{j \in u} h_j^{-2\alpha}$ for all $h$.

Those projection weights are the so-called general weights.
Regrouping by projections: projection weights

Denote \( u(h) = u(h_1, \ldots, h_s) \) the set of indices \( j \) for which \( h_j \neq 0 \). We have

\[
\mathbb{E}[D^2(P_n)] = \sum_{u \subseteq \{1, \ldots, s\}} \sum_{h:u(h) = u} \frac{1}{n} \sum_{i=0}^{n-1} e^{2\pi \sqrt{-1} h^t u_i} \leq \sum_{u \subseteq \{1, \ldots, s\}} D^2_u(P_n).
\]

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\text{Var}[\hat{\mu}_{n, \text{rqmc}}] = \sum_{u \subseteq \{1, \ldots, s\}} \text{Var}[\hat{\mu}_{n, \text{rqmc}}(f_u)].
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Those projection weights are the so-called general weights.

Anyhow, how should we choose them?
Example: a weighted Sobolev space

Space of functions with integrable partial derivatives. RKHS with kernel

\[ K(u, x) = \sum_{u \subseteq \{1, \ldots, s\}} \gamma_u \prod_{j \in u} 2\pi^2 \left[ B_2((u_j - x_j) \mod 1)/2 + (u_j - 0.5)(x_j - 0.5) \right] \]

where \( B_2(u) = u^2 - u + 1/6 \). The shift-invariant kernel is

\[ K_{sh}(u, x) = \sum_{u \subseteq \{1, \ldots, s\}} \gamma_u \prod_{j \in u} 2\pi^2 B_2((u_j - x_j) \mod 1) \]

and the corresponding mean square discrepancy for a randomly-shifted lattice rule with \( v_1 = (v_1, \ldots, v_s) = z/n \) is

\[ \mathbb{E}[D^2(P_n)] = \frac{1}{n} \sum_{i=1}^{n} \sum_{u \subseteq \{1, \ldots, s\}} \gamma_u \prod_{j \in u} 2\pi^2 B_2((i v_j) \mod 1) \].
From the Fourier expansion of $B_2$, we also have

$$\mathbb{E}[D^2(P_n)] = \frac{1}{n} \sum_{i=1}^{n} \sum_{u \subseteq \{1, \ldots, s\}} \gamma_u \prod_{j \in u, h_j \neq 0} h_j^{-2} e^{2\pi \sqrt{-1}ih_jv_j}$$

$$= \sum_{0 \neq h \in L_s^*} \gamma_{u(h)} \prod_{j \in u(h)} h_j^{-2} = \sum_{0 \neq h \in L_s^*} w(h).$$

For those weights, we have $w(h) = |\hat{f}(h)|^2$ for the function

$$f(u) = \sum_{u \subseteq \{1, \ldots, s\}} (2\pi)^{|u|} \gamma_u^{1/2} \prod_{j \in u} (u_j - 0.5),$$

so $\mathbb{E}[D^2(P_n)]$ is the RQMC variance for this $f$.

On the other hand, the ANOVA variance components for this $f$ are

$$\sigma_u^2 = (4\pi^2)^{|u|} \gamma_u \prod_{j \in u} \text{Var}[U - 0.5] = (4\pi^2/12)^{|u|} \gamma_u = (\pi^2/3)^{|u|} \gamma_u.$$

The optimal weights for this $f$ are then

$$\gamma_u = (3/\pi^2)^{|u|} \sigma_u^2 \approx (0.30396)^{|u|} \sigma_u^2.$$
Using the same kernel and a different heuristic argument, Wang and Sloan (2006) come up with weights that generalize to (they do this for product weights only):

$$\gamma_u^2 = \left(\frac{45}{\pi^4}\right) |u| \sigma_u^2,$$

that is,

$$\gamma_u = \left(\sqrt{\frac{45}{\pi^2}}\right) |u| \sigma_u \approx (0.6797) |u| \sigma_u,$$
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With \( \gamma_u = 1 \), we obtain the classical (unweighted) \( P_2 \).
Weighted $\mathcal{P}_{2\alpha}$:

$$\mathcal{P}_{2\alpha} = \sum_{0 \neq h \in L^*_s} \gamma_u(h) (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}$$

Variance for a worst-case function whose square Fourier coefficients are

$$|\hat{f}(h)|^2 = \gamma_u(h) (\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}.$$

This is the RQMC variance for the function

$$f(u) = \sum_{u \subseteq \{1,\ldots,s\}} \sqrt{\gamma_u} \prod_{j \in u} \frac{(2\pi)^\alpha}{\alpha!} B_\alpha(u_j).$$

We also have for this $f$:

$$\sigma_u^2 = \gamma_u \left[ \text{Var}[B_\alpha(U)] \frac{(4\pi^2)^\alpha}{(\alpha!)^2} \right]^{\left|u\right|} = \gamma_u \left[ |B_{2\alpha}(0)| \frac{(4\pi^2)^\alpha}{(2\alpha)!} \right]^{\left|u\right|}.$$
Weighted $\mathcal{P}_{2\alpha}$:

$$\mathcal{P}_{2\alpha} = \sum_{0 \neq h \in L^*} \gamma_u(h)(\max(1, h_1), \ldots, \max(1, h_s))^{-2\alpha}$$

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We also have for this $f$:

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For $\alpha = 1$, we should take $\gamma_u = (3/\pi^2)|u|\sigma_u^2 \approx (0.30396)|u|\sigma_u^2.$

For $\alpha = 2$, we should take $\gamma_u = [45/\pi^4]|u|\sigma_u^2 \approx (0.46197)|u|\sigma_u^2.$

The ratios weight / variance should decrease exponentially with $|u|$. 
Heuristics for choosing the weights

**Idea 1:** take $\gamma_u \approx \sigma_u^2$ or $\gamma_u \approx S_u$ for each $u$. Too simplistic.
Heuristics for choosing the weights

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**Idea 2:** Just take simple order-dependent weights. For example, $\gamma_u = 1$ for $|u| \leq d$ and $\gamma_u = 0$ otherwise. Wang (2007) recommends this with $d = 2$. 
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**Idea 3:** In general, one can define a simple parametric model for the weights and then estimate the parameters by matching the ANOVA variances (e.g., Wang and Sloan 2006).

For example, $\gamma_u = \prod_{j \in u} \gamma_j$ for some constants $\gamma_j \geq 0$ (product weights).

Fewer parameters: take $\gamma_j = a \beta^j$ for $a, \beta > 0$ (geometric).

With a weighted $P_\alpha$-type criterion, we should have $\gamma_u = \rho |u| \sigma_u^2$ for some $\rho > 0$. 

Assume $\gamma_u = \Gamma_{|u|}$. Need to select $\Gamma_1, \ldots, \Gamma_s$.

For each $u$, let $\nu_u^2$ be an estimate of the optimal $\gamma_u$.

Strategy: take $\Gamma_r$ as the average

$$\Gamma_r = \binom{s}{r}^{-1} \sum_{\{u : |u| = r\}} \nu_u^2.$$ 

Here, scaling all weights by the same factor changes nothing.

Ignore one-dimensional projections; they are the same for all lattices.

The idea is to fit the estimated “optimal weights” over all two-dimensional projections via a least-squares procedure. Then we rescale all the weights by a constant factor to match the ratio of average estimated “optimal weights” over the three-dimensional projections to that over the two-dimensional projections.

Let $\tau_j$ be the unscaled weight for projection $j$. We first minimize

$$R = \sum_{k=1}^{s} \sum_{j=1}^{k-1} \left( \tau_j \tau_k - v_{\{j,k\}}^2 \right)^2.$$

Differentiating w.r.t. $\tau_j$ and equaling to 0, we obtain, for each $j$,

$$\tau_j \sum_{k=1, k \neq j}^{s} \tau_k^2 = \sum_{k=1, k \neq j}^{s} \tau_k v_{\{j,k\}}^2.$$
This can be solved by an iterative fixed-point algorithm:

\[
\tau_j^{(0)} = \max_{k,l=1,\ldots,s} \nu_{\{k,l\}}, \quad \tau_j^{(i+1)} = \frac{\sum_{k=1, k \neq j}^{s} \tau_k^{(i)} \nu_{\{j,k\}}^2}{\sum_{k=1, k \neq j}^{s} \left( \tau_k^{(i)} \right)^2},
\]

for \(i = 1, 2, \ldots\).

We then rescale the weights via \(\gamma_j = c \tau_j\) where the constant \(c\) satisfies

\[
\frac{\sum_{k=1}^{s} \sum_{j=1}^{k-1} \tau_j \tau_k}{\sum_{k=1}^{s} \sum_{j=1}^{k-1} \sum_{l=1}^{j-1} \tau_j \tau_k \tau_l} = c \frac{\sum_{k=1}^{s} \sum_{j=1}^{k-1} \nu_{\{j,k\}}^2}{\sum_{k=1}^{s} \sum_{j=1}^{k-1} \sum_{l=1}^{j-1} \nu_{\{j,k,l\}}^2}.
\]
Idea 6: Control the shortest vector in dual lattice, for each projection.

Spectral test for LCGs (Knuth, Fishman, etc.):

\[ \min_{2 \leq r \leq t_1} \frac{\ell_{\{1,\ldots,r\}}}{\ell^*_r(n)} \]

where \( \ell_u \) is the length of a shortest vector in \( L^*_s(u) \) and \( \ell^*_r(n) \) is a theoretical upper bound on this length, in \( r \) dimensions.

Advantages: Computing time of \( \ell_u \) are almost independent of \( n \), although exponential in \( |u| \). Poor lattices can be eliminated quickly: search is fast.
**Idea 6:** Control the shortest vector in dual lattice, for each projection.

Lemieux and L’Ecuyer (2000, etc.) maximize

$$M_{t_1,...,t_d} = \min \left[ \min_{2 \leq r \leq t_1} \frac{\ell_{\{1,\ldots,r\}}}{\ell^*_r(n)}, \min_{2 \leq r \leq d} \min_{u=\{j_1,\ldots,j_r\} \subset \{1,\ldots,s\}} \frac{\ell_u}{\ell^*_r(n)} \right],$$

where $\ell_u$ is the length of a shortest vector in $L^*_s(u)$ and $\ell^*_r(n)$ is a theoretical upper bound on this length, in $r$ dimensions.

Advantages: Computing time of $\ell_u$ are almost independent of $n$, although exponential in $|u|$. Poor lattices can be eliminated quickly: search is fast.

This can of course be generalized by adding weights to projections.
Searching for lattice parameters

Korobov lattices.
Search for $\mathbf{z} = (1, a, a^2, \ldots, \ldots)$ over all admissible integers $a$. 
Searching for lattice parameters

Korobov lattices.
Search for \( z = (1, a, a^2, \ldots, \ldots) \) over all admissible integers \( a \).

Component by component (CBC) construction.

Let \( z_1 = 1 \);

For \( j = 2, \ldots, s \), find \( z_j \in \{1, \ldots, n-1\} \), \( \gcd(z_j, n) = 1 \), such that \((z_1, \ldots, z_{j-1}, z_j)\) minimizes the selected discrepancy for the first \( j \) dimensions.
Searching for lattice parameters

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Search for \( z = (1, a, a^2, \ldots, \ldots) \) over all admissible integers \( a \).

Component by component (CBC) construction.

Let \( z_1 = 1 \);
For \( j = 2, \ldots, s \), find \( z_j \in \{1, \ldots, n - 1\}, \gcd(z_j, n) = 1 \), such that \((z_1, \ldots, z_{j-1}, z_j)\) minimizes the selected discrepancy for the first \( j \) dimensions.

Partial randomized CBC construction.

Let \( z_1 = 1 \);
For \( j = 2, \ldots, s \), try \( r \) random \( z_j \in \{1, \ldots, n - 1\}, \gcd(z_j, n) = 1 \), and retain the one for which \((z_1, \ldots, z_{j-1}, z_j)\) minimizes the selected discrepancy for the first \( j \) dimensions.
Example: stochastic activity network
[Elmaghraby 1977]. Each arc $j$ has random length $V_j = F_j^{-1}(U_j)$.
Let $T = f(U_1, \ldots, U_{13}) = \text{length of longest path from node 1 to node 9}$. 
Want to estimate $q(x) = \mathbb{P}[T > x]$ for a given constant $x$. 

![Stochastic Activity Network Diagram](image-url)
To estimate $q(x)$ by **MC**, we generate $n$ independent realizations of $T$, say $T_1, \ldots, T_n$, and take $(1/n) \sum_{i=1}^{n} \mathbb{I}[T_i > x]$.

For **RQMC**, we replace the $n$ realizations of $(U_1, \ldots, U_{13})$ by the $n$ points of a randomly-shifted lattice.
To estimate $q(x)$ by MC, we generate $n$ independent realizations of $T$, say $T_1, \ldots, T_n$, and take $(1/n) \sum_{i=1}^{n} \mathbb{I}[T_i > x]$.

For RQMC, we replace the $n$ realizations of $(U_1, \ldots, U_{13})$ by the $n$ points of a randomly-shifted lattice.

**Illustration:** $V_j \sim \text{Normal}(\mu_j, \sigma_j^2)$ for $j = 1, 2, 4, 11, 12$, and $V_j \sim \text{Exponential}(1/\mu_j)$ otherwise.

The $\mu_j$: 13.0, 5.5, 7.0, 5.2, 16.5, 14.7, 10.3, 6.0, 4.0, 20.0, 3.2, 3.2, 16.5.
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**Illustration:** $V_j \sim \text{Normal}(\mu_j, \sigma_j^2)$ for $j = 1, 2, 4, 11, 12$, and $V_j \sim \text{Exponential}(1/\mu_j)$ otherwise.

The $\mu_j$: 13.0, 5.5, 7.0, 5.2, 16.5, 14.7, 10.3, 6.0, 4.0, 20.0, 3.2, 3.2, 16.5.

**CMC estimator.** Generate the $V_j$’s only for the 8 arcs that do not belong to the cut $\mathcal{L} = \{5, 6, 7, 9, 10\}$, and replace $\mathbb{I}[T > x]$ by its conditional expectation given those $V_j$’s, $\mathbb{P}[T > x \mid \{V_j, j \not\in \mathcal{L}\}]$.

This makes the integrand continuous in the $U_j$’s.
source

1

2

3

V1

V2

V3

V4

V5

V6

V7

V8

V9

V10

V11

V12

V13

sink
ANOVA Variances for the Stochastic Activity Network

Stochastic Activity Network

- $x = 64$
- $x = 100$
- CMC, $x = 30$
- CMC, $x = 64$
- CMC, $x = 100$

% of total variance

Legend:
- Order 1
- Order 2
- Order 3
- Order 4
- Order 5
- Order 6
- Order 7
ANOVA decomposition

There are six paths from 1 to 9:

\[
\{\{1, 5, 11\}, \{2, 6, 11\}, \{1, 3, 6, 11\}, \{1, 4, 7, 12, 13\}, \{1, 4, 8, 9, 13\}, \{1, 4, 8, 10, 11\}\}. 
\]

Intuition: the important projections should be only the subsets of those paths. Fraction of the total variance that lies in these projections:

<table>
<thead>
<tr>
<th></th>
<th>(x = 30)</th>
<th>(x = 64)</th>
<th>(x = 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>crude MC</td>
<td></td>
<td>80.6 %</td>
<td>96.3 %</td>
</tr>
<tr>
<td>conditional MC</td>
<td>88.8 %</td>
<td>99.5 %</td>
<td>100 %</td>
</tr>
</tbody>
</table>
Lattices of Rank 1 with CBC

Stochastic Activity Network ($x = 64$)

$\text{MC}$

$\text{Sobol}$

$M_{13,13,13,13,13,13,13,13,13,13}$$$

$P_2$ order 2 only

$P_2$ product $v_u^2 = (3/\pi^2) |u| \sigma_u^2$

$P_2$ product $v_u^2 = (45/\pi^4) |u| \sigma_u^2$


$n^{-2}$
Lattices of Rank 1 with CBC

Stochastic Activity Network \((x = 100)\)

\[\text{MC} \]
\[\text{Sobol} \]
\[M_{13,13,13,13,13,13} \]
\[P_2 \text{ order 2 only} \]
\[P_2 \text{ product } v_u^2 = \left(\frac{3}{\pi^2}\right)|u|\sigma_u^2 \]
\[P_2 \text{ product } v_u^2 = \left(\frac{45}{\pi^4}\right)|u|\sigma_u^2 \]
\[P_2 \text{ product Wang & Sloan (2006)} \]
\[n^{-2} \]
Lattices of Rank 1 with CBC

Stochastic Activity Network (CMC \( x = 30 \))

MC
Sobol

\[ M_{13,13,13,13,13,13} \]
weighted \[ M_{13,13,13,13,13,13} \]

\( P_2 \) order 2 only

\[ P_2 \) order \( v_u^2 = \left( \frac{3}{\pi^2} \right) |u| \sigma_u^2 \]

\( P_2 \) product \( v_u^2 = \left( \frac{3}{\pi^2} \right) |u| \sigma_u^2 \)

\( P_2 \) product \( v_u^2 = \left( \frac{3}{\pi^2} \right) |u| \sigma_u^2 \) (no baker)

\( P_2 \) product Wang & Sloan (2006)

\[ n^{-2} \]
Lattices of Rank 1 with CBC

Stochastic Activity Network (CMC $x = 64$)

- MC
- Sobol
  - $M_{13,13,13,13,13,13}$
  - weighted $M_{13,13,13,13,13,13}$
  - $P_2$ order 2 only
    - $P_2$ order $v_u^2 = \frac{3}{\pi^2}|u|\sigma_u^2$
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  - $P_2$ product Wang & Sloan (2006)
  - $n^{-2}$
Lattices of Rank 1 with CBC

Stochastic Activity Network (CMC $x = 100$)

- **MC**
- **Sobol**
  - $M_{13,13,13,13,13,13}$
  - weighted $M_{13,13,13,13,13,13}$
  - $P_2$ order 2 only
  - $P_2$ order $v_u^2 = (3/\pi^2)|u|\sigma_u^2$
  - $P_2$ product $v_u^2 = (3/\pi^2)|u|\sigma_u^2$
  - $P_2$ product $v_u^2 = (3/\pi^2)|u|\sigma_u^2$ (no baker)
  - $P_2$ product $v_u^2 = (45/\pi^4)|u|\sigma_u^2$
  - $P_2$ product Wang & Sloan (2006)

$2^{12}$ to $2^{14}$

$\sigma^2 n = k^2$
Random vs. Full CBC

Stochastic Activity Network (CMC $x = 30$)

- **Full CBC ($\mathcal{P}_2$ product)**
- **Random CBC ($\mathcal{P}_2$ product)**
Random vs. Full CBC

Stochastic Activity Network (CMC $x = 64$)

- $n$ vs. variance
- Full CBC ($P_2$ product)
- Random CBC ($P_2$ product)
Random vs. Full CBC

Stochastic Activity Network (CMC $\times = 100$)

- $\text{Full CBC (P}_2\text{ product)}$
- $\text{Random CBC (P}_2\text{ product)}$

![Graph showing variance against n]
Prime vs. Power-of-2 Number of Points

Stochastic Activity Network (CMC $x = 30$)

- **Prime ($P_2$ product)**
- **Power of 2 ($P_2$ product)**

Variance

$n$

$2^6, 2^8, 2^{10}, 2^{12}, 2^{14}$
Prime vs. Power-of-2 Number of Points

Stochastic Activity Network (CMC $x = 64$)

- Blue line: prime ($P_2$ product)
- Red line: power of 2 ($P_2$ product)

Variation $n$: $2^6, 2^8, 2^{10}, 2^{12}, 2^{14}$
Prime vs. Power-of-2 Number of Points

Stochastic Activity Network (CMC $x = 100$)

- Variance
- $n$ values: $2^6$, $2^8$, $2^{10}$, $2^{12}$, $2^{14}$

**Legend:**
- Blue circles: Prime ($P_2$ product)
- Red squares: Power of 2 ($P_2$ product)
Korobov vs. CBC

Stochastic Activity Network (CMC $x = 30$)

- Solid: CBC.
- Dashed: Korobov.
Korobov vs. CBC

Stochastic Activity Network (CMC $x = 64$)

-variance

Solid: CBC. Dashed: Korobov.
Korobov vs. CBC

Stochastic Activity Network (CMC $x = 100$)

Solid: CBC. Dashed: Korobov.
Histograms, for \( n = 8191 \), \( m = 10^4 \) replications

single MC draw (\( x = 100 \))

MC estimator (\( x = 100 \))

RQMC estimator (\( x = 100 \))
Histograms

single MC draw (CMC $x = 100$)

MC estimator (CMC $x = 100$)

RQMC estimator (CMC $x = 100$)
Function of a Multinormal vector

Let $\mu = E[f(U)] = E[g(Y)]$ where $Y = (Y_1, \ldots, Y_s) \sim N(0, \Sigma)$. 
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For example, if the payoff of a financial derivative is a function of the values taken by a $c$-dimensional geometric Brownian motions (GMB) at $d$ observations times $0 < t_1 < \cdots < t_d = T$, then we have $s = cd$. 
Function of a Multinormal vector

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For example, if the payoff of a financial derivative is a function of the values taken by a \( c \)-dimensional geometric Brownian motions (GMB) at \( d \) observations times \( 0 < t_1 < \cdots < t_d = T \), then we have \( s = cd \).

To generate \( Y \): Decompose \( \Sigma = AA^t \), generate \( Z = (Z_1, \ldots, Z_s) = (\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_s)) \sim N(0, I) \) and return \( Y = AZ \).
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Choice of $A$?
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Choice of $A$?

Cholesky factorization: $A$ is lower triangular.
Principal component decomposition (PCA):
\[ \mathbf{A} = \mathbf{PD}^{1/2} \] where \( \mathbf{D} = \text{diag}(\lambda_s, \ldots, \lambda_1) \) (eigenvalues of \( \mathbf{\Sigma} \) in decreasing order) and the columns of \( \mathbf{P} \) are the corresponding unit-length eigenvectors.
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**Function of a Brownian motion:**
Payoff depends on \( c \)-dimensional Brownian motion \( \{\mathbf{X}(t), t \geq 0\} \) observed at times \( 0 = t_0 < t_1 < \cdots < t_d \).
Principal component decomposition (PCA):
\[ A = PD^{1/2} \] where \( D = \text{diag}(\lambda_s, \ldots, \lambda_1) \) (eigenvalues of \( \Sigma \) in decreasing order) and the columns of \( P \) are the corresponding unit-length eigenvectors. With this \( A \), \( Z_1 \) accounts for the maximum amount of variance of \( Y \), then \( Z_2 \) for the maximum amount of variance conditional on \( Z_1 \), and so on.

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**Sequential (or random walk) method:** generate \( X(t_1) \), then \( X(t_2) - X(t_1) \), then \( X(t_3) - X(t_2) \), etc.
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Brownian bridge (BB) sampling: Suppose \( d = 2^m \). Generate \( X(t_d) \), then \( X(t_d/2) \) conditional on \((X(0), X(t_d))\), then \( X(t_d/4) \) conditional on \((X(0), X(t_d/2))\), and so on.

The first few \( N(0,1) \) r.v.’s already sketch the path trajectory.
Principal component decomposition (PCA): \( \mathbf{A} = \mathbf{P} \mathbf{D}^{1/2} \) where \( \mathbf{D} = \text{diag}(\lambda_s, \ldots, \lambda_1) \) (eigenvalues of \( \Sigma \) in decreasing order) and the columns of \( \mathbf{P} \) are the corresponding unit-length eigenvectors. With this \( \mathbf{A} \), \( Z_1 \) accounts for the maximum amount of variance of \( \mathbf{Y} \), then \( Z_2 \) for the maximum amount of variance conditional on \( Z_1 \), and so on.

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The first few \( N(0,1) \) r.v.'s already sketch the path trajectory.
Each of these methods corresponds to some matrix \( \mathbf{A} \).
Choice has large impact on the ANOVA decomposition of \( f \).
Example: Pricing an Asian option

Single asset, $s$ observation times $t_1, \ldots, t_s$. Want to estimate $\mathbb{E}[f(U)]$, where

$$f(U) = e^{-rt_s} \max \left[ 0, \frac{1}{s} \sum_{j=1}^{s} S(t_j) - K \right]$$

and $\{S(t), t \geq 0\}$ is a geometric Brownian motion. We have $f(U) = g(Y)$ where $Y = (Y_1, \ldots, Y_s) \sim N(0, \Sigma)$. 

Let $S(0) = 100$, $K = 100$, $r = 0.05$, $t_s = 1$, and $t_j = jT/s$ for $1 \leq j \leq s$. We consider $\sigma = 0.2$, 0.5 and $s = 3, 6, 12$. 
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We consider $\sigma = 0.2, 0.5$ and $s = 3, 6, 12$. 

ANOVA Variances for the Asian Option

Asian Option with $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

- $s = 3$, seq.
- $s = 3$, BB
- $s = 3$, PCA
- $s = 6$, seq.
- $s = 6$, BB
- $s = 6$, PCA
- $s = 12$, seq.
- $s = 12$, BB
- $s = 12$, PCA
Asian Option ($s = 6$) with $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$
Asian Option \((s = 6, \text{sequential})\) with \(S(0) = 100, K = 100, r = 0.05, \sigma = 0.5\).
Lattices of Rank 1 with CBC

Asian Option (BB), \( s = 6 \), \( S(0) = 100 \), \( K = 100 \), \( r = 0.05 \), \( \sigma = 0.5 \)

\[ \text{variance} \]

\[ \log_{10} n \]

- MC
- Sobol
- \( M_{6,6,6,6,6,6} \)
- \( \mathcal{P}_2 \) order 2 only
- \( \mathcal{P}_2 \) product \( \nu_u^2 = \sigma_u^2 \)
- \( \mathcal{P}_2 \) product \( \nu_u^2 = (45/\pi^4)|u|\sigma_u^2 \)
- \( \mathcal{P}_2 \) product Wang & Sloan (2006)

\[ n^{-2} \]
Asian Option (PCA) $s = 6$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$. 

$\mathcal{P}_2$ order 2 only

$\mathcal{P}_2$ product $\nu_u^2 = \sigma_u^2$

$\mathcal{P}_2$ product $\nu_u^2 = (45/\pi^4)|u|\sigma_u^2$

$\mathcal{P}_2$ product Wang & Sloan (2006)

$n^{-2}$
Lattices of Rank 1 with CBC

Asian Option (sequential) $s = 12, S(0) = 100, K = 100, r = 0.05, \sigma = 0.5$

$\nu_u^2 = \sigma_u^2$

$\nu_u^2 = (45/\pi^4)|u| \sigma_u^2$


$P_2$ product $P_2$ order 2 only

$M_{12,12,12,12,12,12}$


$n^{-2}$
Asian Option (BB) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

$\mathcal{P}_2$ product $\nu_u^2 = \sigma_u^2$

$\mathcal{P}_2$ product $\nu_u^2 = \left(45/\pi^4\right)|u|\sigma_u^2$

$\mathcal{P}_2$ product Wang & Sloan (2006)

$n^{-2}$
Asian Option (PCA) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

- $\mathcal{M}_{12,12,12,12,12,12,12}$
- $\mathcal{P}_2$ order 2 only
- $\mathcal{P}_2$ product $v_u^2 = \sigma_u^2$
- $\mathcal{P}_2$ product $v_u^2 = (45/\pi^4)|u|\sigma_u^2$
- $\mathcal{P}_2$ product Wang & Sloan (2006)

$n^{-2}$
Random vs. Full CBC

Asian Option (seq.) $s = 12, S(0) = 100, K = 100, r = 0.05, \sigma = 0.5$

![Graph showing variance against n for Full CBC and Random CBC](image-url)
Random vs. Full CBC

Asian Option (BB) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

Graph showing the variance for different values of $n$ with logarithmic scale on the y-axis. The graph compares Full CBC and Random CBC with $P_2$ product.
Random vs. Full CBC

Asian Option (PCA) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$
Asian Option (seq.) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$
Prime vs. Power-of-2 Number of Points

Asian Option (BB) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$
Prime vs. Power-of-2 Number of Points

Asian Option (PCA) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

- Blue dots: prime ($P_2$ product)
- Red squares: power of 2 ($P_2$ product)
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Solid: CBC. Dashed: Korobov.
Asian Option (PCA) $s = 6$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$. 

Solid: CBC. Dashed: Korobov.
Korobov vs. CBC

Asian Option (seq.) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$

Solid: CBC. Dashed: Korobov.
Korobov vs. CBC

Asian Option (BB) \( s = 12, S(0) = 100, K = 100, r = 0.05, \sigma = 0.5\)

Solid: CBC. Dashed: Korobov.
Asian Option (PCA) $s = 12$, $S(0) = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.5$.

Solid: CBC. Dashed: Korobov.
Histograms for the Asian Option, $s = 6$, sequential

single MC draw ($s = 6$, seq.)

MC estimator ($s = 6$, seq.)

RQMC estimator ($s = 6$, seq.)
Histograms for the Asian option, $s = 6$, PCA

single MC draw ($s = 6$, PCA)

MC estimator ($s = 6$, PCA)

RQMC estimator ($s = 6$, PCA)
A down-and-in Asian option with barrier $B$

Same as for Asian option, except that payoff is zero unless

$$\min_{1 \leq j \leq s} S(t_j) \leq 80.$$
ANOVA Variances for the down-and-in Asian Option

Down-and-in with $S(0) = K = 100, r = 0.05, \sigma = 0.2, B = 80$

Order 1  | Order 2  | Order 3  | Order 4  | Order 5  | Order 6  | Order 7
---|---|---|---|---|---|---
$s = 3$, seq.  |  |  |  |  |  |  
$s = 3$, BB  |  |  |  |  |  |  
$s = 3$, PCA  |  |  |  |  |  |  
$s = 6$, seq.  |  |  |  |  |  |  
$s = 6$, BB  |  |  |  |  |  |  
$s = 6$, PCA  |  |  |  |  |  |  
$s = 12$, seq.  |  |  |  |  |  |  
$s = 12$, BB  |  |  |  |  |  |  
$s = 12$, PCA  |  |  |  |  |  |  

% of total variance

% of total variance
Total Variance per Coordinate for the down-and-in Asian Option

Down-and-In ($s = 6$), $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.2$, $B = 80$
Lattices of Rank 1 with CBC

Down-and-In (seq.) $s = 12$, $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $B = 80$

$\text{MC}$

$\text{Sobol}$

$M_{6,6,6,6,6,6}$

$\mathcal{P}_2$ order $\nu^2_u = (45/\pi^4)|u|\sigma^2_u$

$\mathcal{P}_2$ product $\nu^2_u = (45/\pi^4)|u|\sigma^2_u$

$\mathcal{P}_2$ product $\gamma_j = 0.5$

$n^{-2}$

$\text{variance}$
Lattices of Rank 1 with CBC

Down-and-In (BB) $s = 6, S(0) = K = 100, r = 0.05, \sigma = 0.5, B = 80$

\[
\text{MC}
\]
\[
\text{Sobol}
\]
\[
M_{6,6,6,6,6,6}
\]
\[
P_2 \text{ order } v_u^2 = (45/\pi^4)u|u|\sigma_u^2
\]
\[
P_2 \text{ product } v_u^2 = (45/\pi^4)|u|\sigma_u^2
\]
\[
P_2 \text{ product } \gamma_j = 0.5
\]
\[
n^{-2}
\]
Lattices of Rank 1 with CBC

Down-and-In (PCA) \( s = 6, S(0) = K = 100, r = 0.05, \sigma = 0.5, B = 80 \)

\[ P_2 \text{ order } v_u^2 = \left(\frac{45}{\pi^4}\right)|u|\sigma_u^2 \]

\[ P_2 \text{ product } v_u^2 = \left(\frac{45}{\pi^4}\right)|u|\sigma_u^2 \]

\[ P_2 \text{ product } \gamma_j = 0.5 \]

\[ n^{-2} \]
Lattices of Rank 1 with CBC

Down-and-In (seq.) \( s = 12, S(0) = K = 100, r = 0.05, \sigma = 0.5, B = 80 \)

\[
\beta = (45/\pi^4)|u|\sigma_u^2
\]

\( n^{-2} \)
Lattices of Rank 1 with CBC

Down-and-In (BB) $s = 12$, $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $B = 80$
Lattices of Rank 1 with CBC

Down-and-In (PCA) $s = 12$, $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $B = 80$

$P_2$ order $v_u^2 = \left(\frac{45}{\pi^4}\right)|u|\sigma_u^2$

$P_2$ product $v_u^2 = \left(\frac{45}{\pi^4}\right)|u|\sigma_u^2$

$P_2$ product $\gamma_j = 0.5$

$n^{-2}$
Call on the maximum of 6 assets

Each of 6 asset prices obeys a GBM with $s_0 = 100$, $r = 0.05$, $\sigma = 0.2$. The pairwise correlation between Brownian motions is 0.3.

The assets pay a dividend at rate 0.10, which means that the effective risk-free rate can be taken as $r' = 0.05 - 0.10 = -0.05$. 
ANOVA variances for the maximum of 6 assets

Maximum of 6 assets, $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$

Order 1
Order 2
Order 3
Order 4
Order 5
Order 6
Total Variance per Coordinate for max of 6 assets

Maximum of 6 assets, $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$
Lattices of Rank 1 with CBC

Maximum of 6 assets (Cholesky), $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$

\[
\text{variance} = \left(\frac{45}{\pi^4}\right) |u| \sigma_u^2
\]

$P_2$ product $\gamma_j = 0.5$

$n^{-2}$
Lattices of Rank 1 with CBC

Maximum of 6 assets (PCA), $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$
Prime vs. Power-of-2 Number of Points

Maximum of 6 assets with $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$

- Variance
- $n$ values: $2^6$, $2^8$, $2^{10}$, $2^{12}$, $2^{14}$
- Blue dots: prime (P2 order)
- Red squares: power of 2 (P2 order)
Maximum of 12 assets with $S(0) = K = 100$, $r = 0.05$, $\sigma = 0.5$, $\rho = 0.3$.
Discrete choice with multinomial mixed logit probability, max likelihood estimation

Utility of alternative $j$ for individual $q$ is

$$U_{q,j} = \beta_{q}^{t}x_{q,j} + \epsilon_{q,j} = \sum_{\ell=1}^{s} \beta_{q,\ell}x_{q,j,\ell} + \epsilon_{q,j}, \text{ where}$$

$$\beta_{q}^{t} = (\beta_{q,1}, \ldots, \beta_{q,s}) \text{ gives the tastes of individual } q,$$

$$x_{q,j} = (x_{q,j,1}, \ldots, x_{q,j,s}) \text{ attributes of alternative } j \text{ for individual } q,$$

$$\epsilon_{q,j} \text{ noise; Gumbel of mean 0 and scale parameter } \lambda = 1.$$ 

Individual $q$ selects alternative with largest utility $U_{q,j}$.

Can observe the $x_{q,j}$ and choices $y_{q}$, but not the rest.
Logit model: for $\beta_q$ fixed, $j$ is chosen with probability

$$L_q(j \mid \beta_q) = \frac{\exp[\beta_q^t x_{q,j}]}{\sum_{a \in A(q)} \exp[\beta_q^t x_{q,a}]}$$

where $A(q)$ are the available alternatives for $q$. 
Logit model: for $\beta_q$ fixed, $j$ is chosen with probability

$$L_q(j \mid \beta_q) = \frac{\exp[\beta_q^t x_{q,j}]}{\sum_{a \in A(q)} \exp[\beta_q^t x_{q,a}]}$$

where $A(q)$ are the available alternatives for $q$.

For a random individual, suppose $\beta_q$ is random with density $f_{\theta}$, which depends on (unknown) parameter vector $\theta$. We want to estimate $\theta$ from the data (the $x_{q,j}$ and $y_q$).

The unconditional probability of choosing $j$ is

$$p_q(j, \theta) = \int L_q(j \mid \beta) f_{\theta}(\beta) \beta.$$

It depends on $A(q)$, $j$, and $\theta$. 
Maximum likelihood: Maximize the log of the joint probability of the sample, w.r.t. $\theta$:

$$\ln L(\theta) = \ln \prod_{q=1}^{m} p_q(y_q, \theta) = \sum_{q=1}^{m} \ln p_q(y_q, \theta).$$
Maximum likelihood: Maximize the log of the joint probability of the sample, w.r.t. $\theta$:

$$
\ln L(\theta) = \ln \prod_{q=1}^{m} p_q(y_q, \theta) = \sum_{q=1}^{m} \ln p_q(y_q, \theta).
$$

No formula for $p_q(j, \theta)$, but can use MC or RQMC, for each $q$ and fixed $\theta$. Generate $n$ realizations of $\beta$ from $f_\theta$, say $\beta_q^{(1)}(\theta), \ldots, \beta_q^{(n)}(\theta)$, and estimate $p_q(y_q, \theta)$ by

$$
\hat{p}_q(y_q, \theta) = \frac{1}{n} \sum_{i=1}^{n} L_q(j, \beta_q^{(i)}(\theta)).
$$

Then we can find the maximizer $\hat{\theta}$ of $\ln \prod_{q=1}^{m} \hat{p}_q(y_q, \theta)$ w.r.t. $\theta$. 

We take 4 alternatives, with indep. attributes, resp. $N(1, 1), N(1, 1), N(0.5, 1), N(0.5, 1)$. We try $s = 5, 10, 15$. $\beta_q$ is a vector of $s$ indep. $N(1, 1)$ random variables.
Maximum likelihood: Maximize the log of the joint probability of the sample, w.r.t. \( \theta \):

\[
\ln L(\theta) = \ln \prod_{q=1}^{m} p_q(y_q, \theta) = \sum_{q=1}^{m} \ln p_q(y_q, \theta).
\]

No formula for \( p_q(j, \theta) \), but can use MC or RQMC, for each \( q \) and fixed \( \theta \).

Generate \( n \) realizations of \( \beta \) from \( f_{\theta} \), say \( \beta_q^{(1)}(\theta), \ldots, \beta_q^{(n)}(\theta) \), and estimate \( p_q(y_q, \theta) \) by

\[
\hat{p}_q(y_q, \theta) = \frac{1}{n} \sum_{i=1}^{n} L_q(j, \beta_q^{(i)}(\theta)).
\]

Then we can find the maximizer \( \hat{\theta} \) of \( \ln \prod_{q=1}^{m} \hat{p}_q(y_q, \theta) \) w.r.t. \( \theta \).

We take 4 alternatives, with indep. attributes, resp. \( N(1, 1) \), \( N(1, 1) \), \( N(0.5, 1) \), \( N(0.5, 1) \). We try \( s = 5, 10, 15 \).

\( \beta_q \) is a vector of \( s \) indep. \( N(1, 1) \) random variables.
ANOVA Variances for the Mixed Logit Model

Mixed Logit Model

- $s = 5$, individual 1
- $s = 5$, individual 2
- $s = 15$, individual 1
- $s = 15$, individual 2

% of total variance

<table>
<thead>
<tr>
<th>Order</th>
<th>Percent</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>6</td>
<td></td>
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<tr>
<td>7</td>
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</tr>
</tbody>
</table>
Total variance per coordinate

Mixed Logit Model ($s = 5$)

$s = 5$, individual 1

$s = 5$, individual 2

% of total variance
Total variance per coordinate

Mixed Logit Model ($s = 15$)

$s = 15$, individual 1

$s = 15$, individual 2

% of total variance
Lattices of Rank 1 with CBC

Mixed Logit Model \( (s = 10, \text{individual 1}) \)

\[ \text{MC} \]

\[ \text{Sobol} \]

\[ M_{10,10,10,10,10,10} \]

\[ \mathcal{P}_2 \text{ order } v^2_u = (45/\pi^4)|u|\sigma^2_u \]

\[ \mathcal{P}_2 \text{ product } v^2_u = (45/\pi^4)|u|\sigma^2_u \]

\[ \mathcal{P}_2 \text{ product Wang & Sloan (2006)} \]

\[ \mathcal{P}_2 \text{ product } \gamma_j = 0.5 \]

\[ n^{-2} \]
Lattices of Rank 1 with CBC

Mixed Logit Model ($s = 10$, individual 2)

$\text{MC}$

$\text{Sobol}$

$M_{10,10,10,10,10,10,10}$

$P_2$ order $v_u^2 = \left(\frac{45}{\pi^4}\right)u|\sigma_u^2$

$P_2$ product $v_u^2 = \left(\frac{45}{\pi^4}\right)u|\sigma_u^2$


$P_2$ product $\gamma_j = 0.5$

$n^{-2}$
Lattices of Rank 1 with CBC

Mixed Logit Model ($s = 15$, individual 1)

\[ \text{MC} \]
\[ \text{Sobol} \]
\[ M_{15,15,15,15,15,15} \]
\[ \mathcal{P}_2 \text{ order } v_u^2 = \left( \frac{45}{\pi^4} \right) |u| \sigma_u^2 \]
\[ \mathcal{P}_2 \text{ product } v_u^2 = \left( \frac{45}{\pi^4} \right) |u| \sigma_u^2 \]
\[ \mathcal{P}_2 \text{ product Wang & Sloan (2006)} \]
\[ \mathcal{P}_2 \text{ product } \gamma_j = 0.5 \]
\[ n^{-2} \]
Lattices of Rank 1 with CBC

Mixed Logit Model \((s = 15, \text{individual 2})\)

\[\gamma_j = 0.5\]

\[n^{-2}\]
Random vs. Full CBC

Mixed Logit Model \((s = 10, \text{ individual 1})\)

- **Full CBC** \((P_2\ \text{order})\)
- **Random CBC** \((P_2\ \text{order})\)
Random vs. Full CBC

Mixed Logit Model ($s = 10$, individual 2)
Prime vs. Power-of-2 Number of Points

 Mixed Logit Model ($s = 10$, individual 1)

 variance
 $10^{-4}$ $10^{-5}$ $10^{-6}$ $10^{-7}$

 $n$

 $2^6$ $2^8$ $2^{10}$ $2^{12}$ $2^{14}$

- prime ($P_2$ order)
- power of 2 ($P_2$ order)
Prime vs. Power-of-2 Number of Points

Mixed Logit Model ($s = 10$, individual 2)

- prime ($\mathcal{P}_2$ order)
- power of 2 ($\mathcal{P}_2$ order)
Korobov vs. CBC

Mixed Logit Model \((s = 10, \text{individual 1})\)

\[ M_{32,24,16,12}, M_{10,10,10,10,10,10}, \mathcal{P}_2 \text{ order} \]

Solid: CBC. Dashed: Korobov.
Korobov vs. CBC

Mixed Logit Model ($s = 10$, individual 2)

Solid: CBC. Dashed: Korobov.