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Exact Simulation Methods for Pricing Occupation-Time Derivatives

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MCQMC 2010
Warsaw, August 15–20, 2010

Agenda

- Option Pricing under a Solvable Diffusion
- Occupation-Time Derivatives
- Known Results for Brownian Motion
- Exact Simulation of Bessel Processes
- The CEV Asset Pricing Model

Pricing under a Solvable Diffusion Model

- Consider a financial market with
  - a money market account $(B_t = B_0 \mathrm{e}^{rt})_{t \geq 0}$;
  - a risky asset with price process $(S_t)_{t \geq 0} \in R_+$.

- Let the asset price process $(S_t)_{0 \leq t \leq T}$ be governed (under a risk-neutral measure) by an SDE

$$dS_t = rS_t dt + \sigma(S_t) dW_t$$

- Assume that the transition PDF is available and the precise path sampling of the process (S_t) is possible.
- That is, for every choice of time-partition $0 = t_0 < t_1 < \cdots < t_n$, we are able to sample the skeleton $(S_{t_0}, S_{t_1}, \ldots, S_{t_n})$ from its exact multivariate probability distribution.

Motivation: Pricing (Path-Dependent) Derivatives

Example: A call/put option is a financial contract that allows its holder to buy/sell the asset under some specified in advance conditions.

European-type option can be exercised at expiry only.

- Pricing is reduced to computing (path-)integrals:

$$V_0 = \mathbb{E}_Q \left[\frac{B_T}{B_0} V_T \right] = \frac{B_T}{B_0} \int_D V_T(\mathbf{S}) p(\mathbf{S})\, d\mathbf{S}.$$

- Use (Q)MCM for computing multidimensional integrals.
- In general, sampling from the exact distribution of (S_t) is not required thanks to the important sampling principle

American-type option can be exercised before expiry.

- Pricing is reduced to solving a dynamic-programming problem

$$V_0 = \sup_{\tau} \mathbb{E}_Q \left[\frac{B_\tau}{B_0} V_\tau(S_\tau) \right], \text{ where } \tau \text{ is a stopping time.}$$

- Use (Q)MCM with the SMM or LS regression method
- Have to sample paths from the exact distribution of (S_t)

Occupation Time and Quantiles

- The occupation time of the process (S_t) staying *below* the barrier L is

$$A_T^{L,-}(S) = \int_0^T 1_{S_t \le L} dt$$

- The occupation time of $(S_t)_{0 \le t \le T}$ staying *above* the barrier L is

$$A_T^{L,+}(S) = \int_0^T 1_{S_t \ge L} dt = T - A_T^{L,-}(S)$$

- The α-quantile ($0 \le \alpha \le 1$) is defined by

$$Q_T^{\alpha}(S) = \inf\{L \in R \,:\, A_T^{L,-}(S) > \alpha T\}$$

Class of Occupation Time Derivatives

- Consider a contract with a payoff of the form

$$\Lambda(A_T^{L,\pm}(S), S_T) = f(A_T^{L,\pm})g(S_T)$$

- Examples include:
  - Step options with payoff $\exp(-\rho A_T^{L,\pm})g(S_T)$
  - Cumulative Parisian options with payoff $1_{A_T^{L,\pm} < \alpha T} g(S_T)$
- Other Occupation-time Options:
  - Quantile options with payoff $\Lambda(Q_T^{\alpha}(S), S_T)$
  - Parisian options

Quantile Options (Miura, 1992)

- The payoff of a α-quantile option depends on the α-quantile of the asset price process S
- The terminal payoff functions of the fixed strike call and floating strike put α-quantile options are respectively defined by

$$\left(Q_T^\alpha(S) - K\right)_+ \text{ and } \left(Q_T^\alpha(S) - S_T\right)_+$$

- Relation to standard lookback options:

$$\lim_{\alpha\to 0} Q_T^\alpha(S) \overset{\text{a.s.}}{=} \inf_{0\le t\le T} S_t \triangleq m_T(S) \quad \lim_{\alpha\to 1} Q_T^\alpha(S) \overset{\text{a.s.}}{=} \sup_{0\le t\le T} S_t \triangleq M_T(S)$$

Step Options (Linetsky, 1999)

- Introduce a finite *knock-out-rate* $\rho \in [0, \infty]$ and a pre-specified price level (barrier) L
- Define the payoff of step call and put options with strike K

$$\Lambda = \exp(-\rho A_T^{L,+}(S))(S_T - K)_+ \quad \text{up-and-out call}$$
$$\Lambda = \exp(-\rho A_T^{L,-}(S))(K - S_T)_+ \quad \text{down-and-out put}$$

- Relation to standard barrier options:

$$\lim_{\rho\to\infty} \exp(-\rho A_T^{L,+}(S)) \stackrel{\text{a.s.}}{=} 1_{M_T(S)<L}$$
$$\lim_{\rho\to\infty} \exp(-\rho A_T^{L,-}(S)) \stackrel{\text{a.s.}}{=} 1_{m_T(S)>L}$$

- Relation to standard vanilla options:

$$\lim_{\rho\to 0} \exp(-\rho A_T^{L,\pm}(S)) = 1 \;\Rightarrow\; \Lambda = g(S_T)$$

Known Results for the case of Brownian Motion

- The joint distribution of $(A_T^{L,\pm}, S_T)$ can be obtained by solving the Feynman-Kac problem followed by analytical inversion of the double Laplace transform
- Dassios (1995) shows for Brownian Motion that

$$Q_T^{\alpha}(B) \text{ and } \sup_{0\le s\le \alpha T} B_s + \inf_{0\le s\le (1-\alpha)T} \widetilde{B}_s$$

  are equal in law, where B and $\widetilde{B}$ are independent Brownian motions
- In case of a geometric Brownian motion, analytical formulae for step, quantile, and Parisian options are available

Occupation Time of Brownian Bridge

- Consider Brownian bridge $(B_t^{0,x})_{0\le t\le 1}$ over $[0,1]$ from $B_0=0$ to $B_1=x$.
- The probability distribution of the occupation time above $\ell\ge 0$, $A_1^{\ell,+}(B)$, is
- Case of $x\le\ell$:

$$\mathbb{P}(A_1^{\ell,+}\le\tau)=1-\frac{2\sqrt{\tau}}{\sqrt{\pi}}e^{\frac{x^2}{2}}\int_\tau^1 e^{-\frac{(2\ell-x)^2}{2(1-u)}}\frac{\sqrt{u-\tau}}{\sqrt{\pi(1-u)}u^2}\,du$$

- Case of $x>\ell$:

$$\mathbb{P}(A_1^{\ell,+}\le\tau)=\int_0^\tau\frac{(\tau-u)e^{\frac{x^2}{2}-\frac{\ell^2}{2(1-u)}-\frac{(x-\ell)^2}{2u}}}{\sqrt{2\pi}\,(u(1-u))^{\frac{3}{2}}}\left(\frac{\ell(x-\ell)^2}{u}-\frac{(x-\ell)\ell^2}{1-u}+x-2\ell\right)du$$

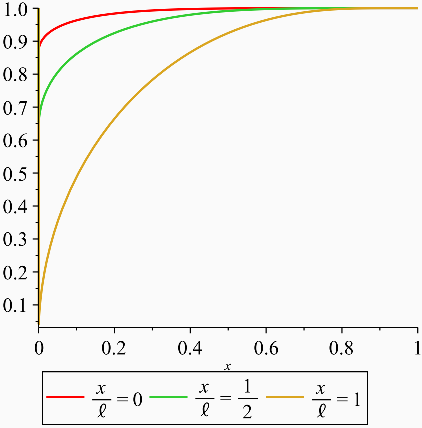
- For an arbitrary time interval of length T we have that

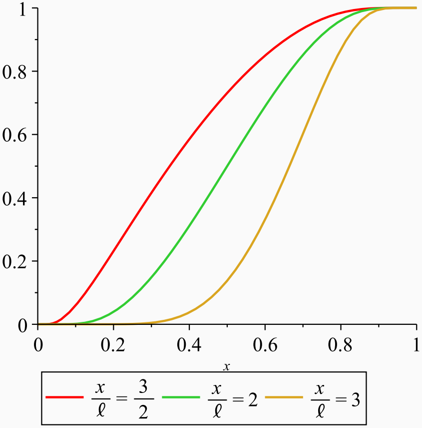
$$\mathbb{P}\left(A_T^{\ell,+}(B)\le\tau\mid B_0=0,B_T=x\right)=\mathbb{P}\left(A_1^{\frac{\ell}{\sqrt{T}},+}(B)\le\frac{\tau}{T}\mid B_0=0,B_1=\frac{x}{\sqrt{T}}\right)$$

  thanks to the scaling property of Brownian motion

Here we present extended results of Hooghiemstra, 2002.

CDF of the Occupation Time of a Brownian Bridge





Simulation of Occupation Time of Brownian Bridge

- Case of $x \le \ell$ (ends below or at the level):
  - Draw the occupation time $A_T^{\ell,+}(B^{0,x})$ of the Brownian bridge over $[0, T]$ from 0 to x.
- Case of $x > \ell$ (ends above the level):
  - First, draw the FHT $\tau_\ell(B^{0,x}) \in (0, T)$ at barrier $x = \ell$.
  - Second, draw the occupation time $A_{T-\tau_\ell}^{y,+}(B^{0,y})$ of the Brownian bridge over $[0, T - \tau_\ell]$ from 0 to $y = x - \ell$.
- The advantage of such a scheme is that the CDFs of both $A_T^{\ell,+}(B^{0,x})$, $x \le \ell$, and $\tau_\ell(B^{0,x})$, $x > \ell$, admit representations in terms of the error function (i.e., in terms of the standard normal CDF). One can use the standard inverse CDF method.

Brownian Bridge Interpolation of a Solvable Diffusion

- Consider a solvable diffusion driven by SDE

$$dS_t = rS_t dt + \sigma(S_t) dW_t, \quad S_{t=0} = S_0$$

- Apply a change of variables $X_t = \mathrm{X}(S_t)$ to obtain

$$dX_t = b(S_t) dt + dW_t, \quad X_{t=0} = \mathrm{X}(S_0)$$

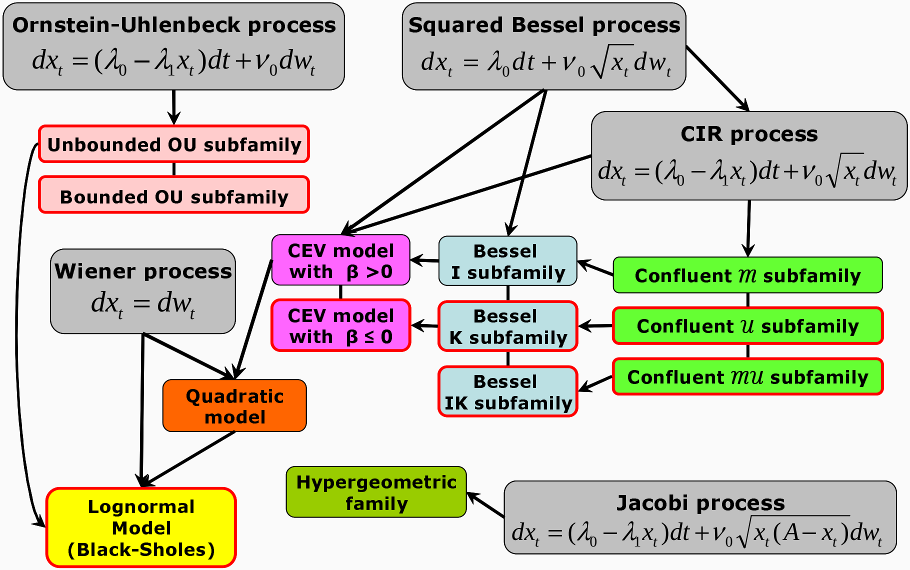
- The mapping is given by $\mathrm{X}(s) = \bar{x} + \int_{\bar{s}}^{s} \frac{1}{\sigma(u)} \, du$
- Consider a time partition $0 = t_0 < t_1 < \cdots < t_N = T$
- Sample an exact skeleton $(X_{t_i})_{i=0,1,\ldots,N}$
- Interpolate with independent Brownian bridges:

$$(\widetilde{X}_t)_{0 \le t \le T} \ : \ \forall t \in [t_{i-1}, t_i] \ \widetilde{X}_t \stackrel{d}{=} \{B_t \mid B_{t_{i-1}} = X_{t_{i-1}}, B_{t_i} = X_{t_i}\}$$

Approximate Simulation of Occupation Time $A_T^L(S)$

- Reduce the SDE for (S_t) to one with unit diffusion coefficient by using a change of variables $X_t = \mathrm{X}(S_t)$
- Draw a skeleton $(X_{t_1}, \ldots, X_{t_N})$ from the exact distribution
- For each $i = 1, \ldots, N$ simulate the occupation time $A_i \triangleq A^{\ell}_{t_i - t_{i-1}}(B)$ of Brownian bridge over $[0, t_i - t_{i-1}]$ from $X_{t_{i-1}}$ to X_{t_i}, where $\ell = \mathrm{X}(L)$
- Set $A_T^L(S) \approx A_T^{\ell}(\widetilde{X}) = \sum_{i=1}^{N} A_i$
- Return the approximate value of $A_T^L(S)$ and the *exact* skeleton $(S_{t_i} = \mathrm{F}(X_{t_i}),\ i = 1, \ldots, N)$, where $\mathrm{F} \triangleq \mathrm{X}^{-1}$.

A sample sub-Hierarchy of Exactly Solvable Kernels



Transition and Bridge PDFs

- Consider a continuous-time time-homogeneous stoch. process $(S_t)_{t\geq 0}$
- The transition PDF $p(t; S_0, S)$ of the process (S_t) is defined by

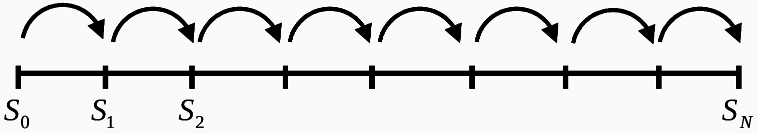
$$p(t; s_0, s)ds = \mathbb{P}(S_t \in (s, s + ds) \mid S_0 = s_0).$$

- Let $0 \leq t_1 < t < t_2$. Consider a stochastic bridge process generated by (S_t) with S_{t_1} and S_{t_2} tied at s_1 and s_2 respectively.
- The bridge PDF $p_{s_1,s_2}(t; s) \equiv p_{(t_1;s_1),(t_2;s_2)}(t; s)$ defined by $p_{s_1,s_2}(t; s)ds = \mathbb{P}(S_t \in (s, s + ds) \mid S_{t_1} = s_1, S_{t_2} = s_2)$ is given by

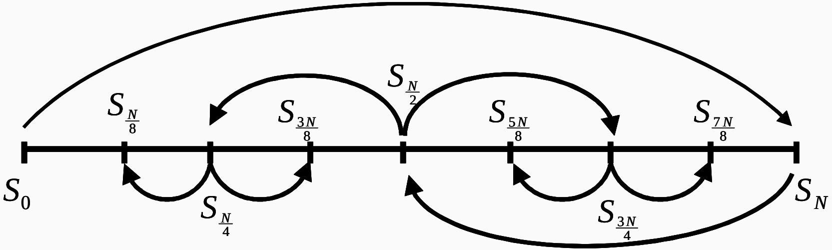
$$p_{(t_1;s_1),(t_2;s_2)}(t; s) = \frac{p(t - t_1; s_1, s)p(t_2 - t; s, s_2)}{p(t_2 - t_1; s_1, s_2)}$$

- For Gaussian processes one may also use a conditional multivariate normal distribution to obtain the bridge PDF in closed form.

Sequential Sampling vs. Bridge Sampling



$$p(\mathbf{S}) = p(\tfrac{T}{N}; S_0, S_1,)\, p(\tfrac{T}{N}; S_1, S_2) \cdots p(\tfrac{T}{N}; S_{N-1}, S_N)$$



$$p(\mathbf{S}) = p(T; S_0, S_N)\, p_{S_0,S_N}(\tfrac{T}{2}; S_{\frac{N}{2}})\, p_{S_0,S_{\frac{N}{2}}}(\tfrac{T}{4}; S_{\frac{N}{4}})\, p_{S_{\frac{N}{2}},S_N}(\tfrac{3T}{4}; S_{\frac{3N}{4}}) \cdots$$

Modelling Processes with Absorption

- Assume that stochastic process $(S_t)_{t\geq 0} \in R_+$ admits absorption at the origin. Let the first hitting time PDF $q(S_0; t)$ be available.
- The transition PDF p does not integrate to one. Define the probabilities of surviving and absorption before time t:

$$P_s(s_0; t) = \int_0^\infty p(t; s_0, s)ds \text{ and } P_a(s_0; t) = 1 - P_s(s_0; t) > 0$$

- The actual transition distribution is a mixture of continuous and discrete probability distributions:

$$p(S_0 \to S_t) = P_s(S_0; t) \cdot \left(\frac{p(t; S_0, S_t)}{P_s(S_0; t)} \right) + P_a(S_0; t) \cdot \delta(S_t)$$

- The PDF $q(S_0; \tau)$ of the first hitting time (FHT) $\tau_0 = \inf\{t \ : \ S_t = 0\}$ is

$$q(S_0; \tau) = \frac{\partial}{\partial \tau} P_a(S_0; \tau) = -\frac{\partial}{\partial \tau} \int_0^\infty p(\tau; S_0, S)dS$$

Bessel Squared Process $BESQ^{(\mu)}(x)$

Consider a squared Bessel process $(X_t)_{t\geq 0} \in (0, \infty)$

$$dX_t = \lambda dt + \nu\sqrt{X_t}dW_t,\ t > 0, \quad X_0 = x$$

with the transition PDF

$$p(t; x, y) = \frac{1}{\nu^2 t/2}\left(\frac{y}{x}\right)^{\mu/2} \exp\left(-\frac{x+y}{\nu^2 t/2}\right) I_{\tilde{\mu}}\left(\frac{\sqrt{xy}}{\nu^2 t/4}\right), \quad t > 0, \quad x, y > 0,$$

where λ is called the dimension and $\mu = 2\lambda/\nu^2 - 1$ is called the index.

The Boundary Classification at the Origin

$\mu \geq 0$: 0 is entrance-not-exit ($\tilde{\mu} = \mu$)

$\mu \in (-1, 0)$: 0 is regular. Let $\tilde{\mu} = \mu$, then 0 is reflecting; Let $\tilde{\mu} = |\mu|$, then 0 is killing

$\mu \leq -1$: 0 is exit-not-entrance ($\tilde{\mu} = |\mu|$)

Sequential Sampling Algorithm (without Absorption)

- Let $\mu > -1$ and $\tilde{\mu} = \mu$. So the origin is an entrance or regular reflecting boundary

input $X_0 > 0$, $0 = t_0 < t_1 < \cdots < t_N$
for $n = 1$ to N **do**

$$Y_n \sim \mathrm{P}\left(\frac{X_{n-1}}{2(t_n - t_{n-1})}\right)$$

$$X_n \sim \mathrm{G}\left(Y_n + \mu + 1, \frac{1}{2(t_n - t_{n-1})}\right)$$

end for
return $(X_0, X_1, \ldots, X_N)$

The FHT distribution of $BESQ^{(\mu)}(x)$

- Let $\mu < 0$ and $\tilde{\mu} = |\mu|$. So the origin is an exit or regular killing boundary
- The PDF q of the FHT $\tau_0 = \inf\{t \,:\, X_t = 0 \mid X_0 = x\}$ is

$$q(x;\tau) = \frac{1}{\Gamma(|\mu|)\,\tau}\left(\frac{x}{2\tau}\right)^{|\mu|} \exp\left(-\frac{x}{2\tau}\right)$$

- Sampling formula: $\tau_0 = \frac{x}{2Y}$, where $Y \sim \mathrm{Gamma}(|\mu|, 1)$
- The probabilities of surviving and absorption before time t are

$$P_s(x;t) = \mathbb{P}(\tau_0 > t) = \frac{\gamma\left(|\mu|, \frac{2x}{\nu^2 t}\right)}{\Gamma(|\mu|)} \text{ and } P_a(x;t) = \mathbb{P}(\tau_0 \le t) = \frac{\Gamma\left(|\mu|, \frac{2x}{\nu^2 t}\right)}{\Gamma(|\mu|)}$$

Notice: For simplicity of presentation, we assume here that $\nu = 2$. A simple scale transformation $X_t^{(\nu_1,\lambda_1)} = \left(\frac{\nu_1}{\nu_2}\right)^2 X_t^{(\nu_2,\lambda_2)}$, $\lambda_1 = \lambda_2\left(\frac{\nu_1}{\nu_2}\right)^2$, $\mu_1 = \mu_2$, allows us to modify ν.

Sequential Sampling Algorithm

- Consider the sequential sampling $x_0 \to x_1 \to x_2 \to \cdots \to x_n$, $x_i := X_{t_i}$ for $0 = t_0 < t_1 < t_2 < \cdots < t_n$ with $\Delta t_i \equiv t_{i+1} - t_i$
- The actual transition distribution is

$$p(x \to y) = P_s(x; \Delta t) \cdot \left(\frac{p(\Delta t; x, y)}{P_s(x; \Delta t)} \right) + P_a(x; \Delta t) \cdot \delta(y)$$

- The normalized distribution is a mixture distribution

$$\frac{p(\Delta t; x, y)}{P_s(x; \Delta t)} = \sum_{n=0}^{\infty} e^{-\frac{x}{2\Delta t}} \frac{\left(\frac{x}{2\Delta t}\right)^{n+|\mu|}}{\Gamma(n+|\mu|+1)} \frac{\Gamma(|\mu|)}{\gamma\left(|\mu|, \frac{x}{2\Delta t}\right)} \frac{y^n \mathrm{e}^{-\frac{y}{2\Delta t}}}{n!\,(2\Delta t)^{n+1}}$$

- It is a mixture of the Gamma probability distribution and “incomplete Gamma” discrete probability distributions $\mathrm{I}\Gamma$
- The transition PDF of the squared Bessel process with absorption follows a randomized Gamma distribution $G(\gamma + 1, 1/(2t))$ where $\gamma \sim \mathrm{I}\Gamma(x/(2\Delta t), |\mu|)$.
- The incomplete Gamma distribution is log-concave and unimodal

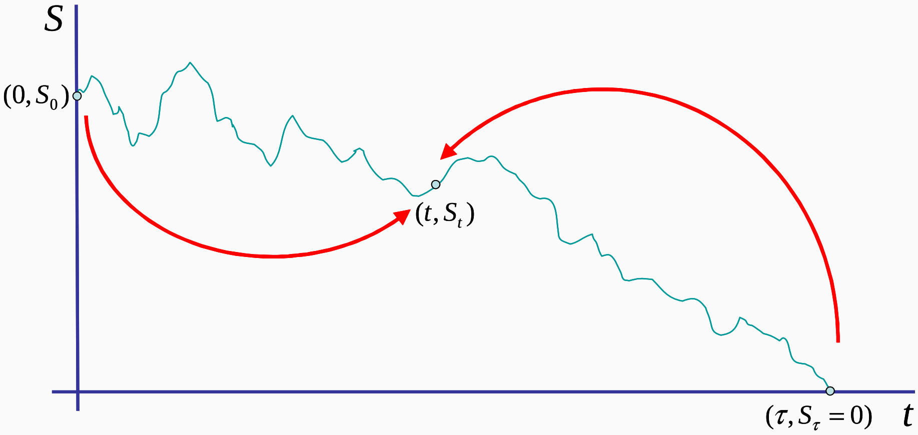
Sequential Sampling Algorithm (with Absorption)

input $X_0 > 0$, $0 = t_0 < t_1 < \cdots < t_N$, $\mu < 0$
$\tilde{\tau}_0 \leftarrow \infty$
for $n = 1$ to N **do**
  if $\tilde{\tau}_0 = \infty$ **then**
    $p_a \leftarrow \Gamma\left(|\mu|, \frac{X_{n-1}}{2(t_n - t_{n-1})}\right) / \Gamma(|\mu|)$ and $U_n \sim \mathrm{U}(0,1)$
    if $U_n < p_a$ **then** $\tilde{\tau}_0 \leftarrow t_n$
  end if
  if $t_n < \tilde{\tau}_0$ **then**
    $Y_n \sim \mathrm{I\Gamma}\left(|\mu|, \frac{X_{n-1}}{2(t_n - t_{n-1})}\right)$ and $X_n \sim \mathrm{G}\left(Y_n + 1, \frac{1}{2(t_n - t_{n-1})}\right)$
  else
    $X_n \leftarrow 0$
  end if
end for
return $(X_0, X_1, \ldots, X_N)$ and $\tilde{\tau}_0 \in \{1, 2, \ldots, N, \infty\}$

Main Idea: Sampling Conditional on the FHT

Step 1 Sample the first hitting time τ (from the FHT PDF $p(S_0;\tau)$)

Step 2 Sample the stochastic process conditional on S_0 and $S_\tau = 0$ (using the distribution of the bridge process)



The distribution of $BESQ_T^{(\mu)}(x,0)$

$BESQ_T^{(\mu)}(x,y)$: The bridge PDF $p_{(0;x),(T;y)}(t;z)$ of the squared Bessel bridge process X_t, $0 \le t \le T$, conditional on $X_0 = x$ and $X_T = y$ is

$$p_{x,y}(t;z) = \frac{p(t;x,z)p(T-t;z,y)}{p(T;x,y)}$$
$$= \frac{T}{2t(T-t)} e^{-\dfrac{\bar{x}+\bar{z}}{2t} - \dfrac{\bar{y}t}{2}} \frac{I_{\tilde{\mu}}(\sqrt{\bar{x}\bar{z}}/t) I_{\tilde{\mu}}(\sqrt{\bar{z}\bar{y}}/(T-t))}{I_{\tilde{\mu}}(\sqrt{\bar{x}\bar{y}}/T)}$$

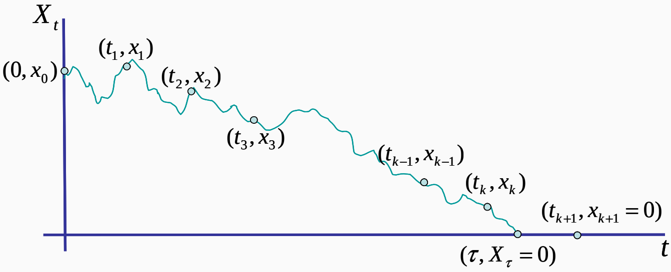
where $\bar{x} := \frac{x\,(T-t)}{T}$, $\bar{z} := \frac{z\,T}{T-t}$, and $\bar{y} := \frac{y}{T(T-t)}$

$BESQ_T^{(\mu)}(x,0)$: In the limiting case as $y \to 0+$, we obtain

$$p_{x,0}(t;z) = \frac{T}{2t(T-t)} \left(\frac{\bar{z}}{\bar{x}}\right)^{\tilde{\mu}/2} \exp\left(-\frac{\bar{x}+\bar{z}}{2t}\right) I_{\tilde{\mu}}\left(\frac{\sqrt{\bar{x}\bar{z}}}{t}\right)$$

Bridge Sampling Algorithm Conditional on the FHT

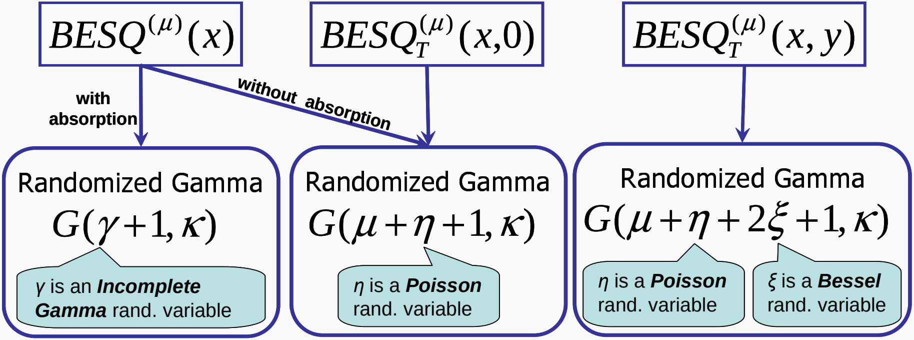
- Consider the sequential sampling $x_0 \to x_1 \to x_2 \to \cdots \to x_n$, $x_i := X_{t_i}$ for $0 = t_0 < t_1 < t_2 < \cdots < t_n$ with $\Delta t_i \equiv t_{i+1} - t_i$
- Sample the FHT $\tau = \dfrac{x_0}{2Y}$, where $Y \sim \mathrm{Gamma}(|\mu|, 1)$
- For all $i > 0$ s.t. $t_i < \tau$, sample x_i conditional on $X_{t_i} = x_{i-1}$ and $X_\tau = 0$ using the randomized gamma with a Poisson randomizer
- For all $i > 0$ s.t. $\tau \le t_i$, set $x_i = 0$



Bridge Sampling Algorithm

input $X_0 > 0$, $0 = t_0 < t_1 < \cdots < t_N$, $\mu < 0$
$Y \sim \mathrm{G}(|\mu|, 1)$
$\tau_0 \leftarrow \dfrac{X_0}{2Y}$
for $n = 1$ to N **do**
  if $t_n < \tau_0$ **then**
    $Y_n \sim \mathrm{P}\left(\dfrac{X_{n-1}(\tau_0 - t_n)}{2(\tau_0 - t_{n-1})(t_n - t_{n-1})}\right)$
    $X_n \sim \mathrm{G}\left(Y_n + |\mu| + 1, \dfrac{\tau_0 - t_{n-1}}{(\tau_0 - t_n)(t_n - t_{n-1})}\right)$
  else
    $X_n \leftarrow 0$
  end if
end for
return $(X_0, X_1, \ldots, X_N)$ and τ_0

Simulation with Randomized Gamma Distributions



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- L. Devroye, Simulating Bessel random variables, *Statistics and Probability Letters*, 57 (2002) 249–257

Randomizers

The Poisson Distribution with mean λ:

$$\mathbb{P}(\eta = n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, 2, \ldots$$

The Bessel Distribution with parameters $a > 0$ and $\mu > -1$:

$$\mathbb{P}(\xi = n) = \frac{(a/2)^{2n+\mu}}{I_\mu(a)\; n!\; \Gamma(n+\mu+1)}, \quad n = 0, 1, 2, \ldots$$

This expression follows from $I_\mu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\mu}}{n!\;\Gamma(n+\mu+1)}$.

The Incomplete Gamma Distribution parameters $a > 0$ and $\mu > -1$:

$$\mathbb{P}(\gamma = n) = \frac{a^{n+\mu} e^{-a}}{\Gamma(n+\mu+1)} \frac{\Gamma(\mu)}{\gamma(\mu, a)}, \quad n = 0, 1, 2, \ldots$$

If $\mu = 0, 1, 2, \ldots$, then it is a truncated and shifted Poisson distribution thanks to $\frac{\gamma(m,a)}{\Gamma(m)} = 1 - \left(1 + x + \ldots + \frac{x^{m-1}}{(m-1)!}\right) e^{-x}$

Sampling Randomized Gammas

- The Poisson, Bessel or incomplete gamma probability distributions are all log-concave (i.e. $\mathbb{P}\{Y = n + 1\}/\mathbb{P}\{Y = n\}$ is decreasing in n) and unimodal.
- To generate a Bessel or incomplete gamma random variate, we can use a generic acceptance-rejection (A-R) method by Devroye.
- As an alternative sampling method we use the inverse method (I-M) by chop-down search from the mode m.
- The chop-down search sampling method is based on the numerical inversion of the CDF F:

$$F^{-1}(u) = \arg\min\{n \geq 0 \mid u - \sum_{k=0}^{n} p_k < 0\},\ u \in [0, 1].$$

- It is well known that the computational cost of this method has the lowest possible value if and only if the vector of discrete probabilities is arranged in increasing order.
- We start the search algorithm at the mode m and then successively (and recursively) calculate probabilities of values to the left and to the right of the mode choosing the largest one.

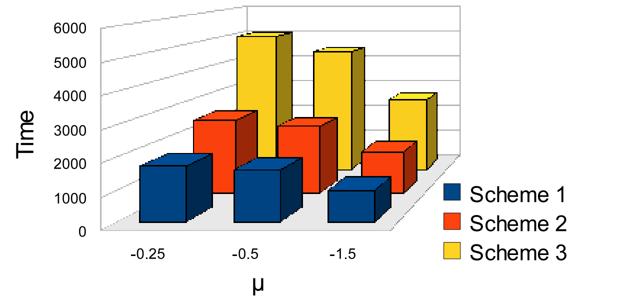
Comparison

Table: of the acceptance-rejection and inverse-CDF methods for the Poisson, Bessel, and incomplete gamma distributions.

Test No.	Distribution	Method	Time (sec)	Avg. # of Steps
1	P(λ)	A-R	189.9	2.6
	λ ~U(0,1)	I-M	35.2	34.2
2	Bes(θ,b)	A-R	220.6	1.6
	θ ~U(0,1000), $b = 10$	I-M	100.4	1.1
3	Bes(θ,b)	A-R	414.1	4.0
	$\theta = 10$, b ~U(0,1000)	I-M	103.3	17.3
4	I$\Gamma(\theta, \lambda)$	A-R	336.6	3.7
	θ ~U(0,100), λ =10	I-M	51.1	1.8
5	I$\Gamma(\theta, \lambda)$	A-R	363.7	3.9
	$\theta = 10, \lambda$ ~U(0,100)	I-M	51.4	10.8

Simulation of the SQB Process

- In this section we aim to compare the following three sampling schemes:
  1. Sequential sampling conditional on the FHT, τ_0, with the use of the randomized gamma distribution of the first kind;
  2. Unconditional sequential sampling with the use of the randomized gamma distribution of the third kind;
  3. Full bridge sampling conditional on the FHT, τ_0, with the use of the randomized gamma distribution of the second kind.
- We compare the times required to simulate 10^6 paths with $N = 32$ points.



The Cox-Ingersoll-Ross model

Consider the CIR process $(X_t)_{t\geq 0} \in (0,\infty)$

$$dX_t = (\lambda_0 - \lambda_1 X_t)dt + \nu\sqrt{X_t}dW_t\,, \quad t > 0,$$

Denote $c_t \equiv \kappa/(e^{\lambda_1 t} - 1)$. The transition PDF is

$$p(t; x_0, x) = c_t e^{\lambda_1 t}\left(\frac{xe^{\lambda_1 t}}{x_0}\right)^{\frac{\mu}{2}} e^{-c_t(xe^{\lambda_1 t}+x_0)} I_{\tilde{\mu}}\left(2c_t\sqrt{xx_0e^{\lambda_1 t}}\right)\,,$$

The Boundary Classification at the Origin

$\mu \geq 0$: 0 is entrance-not-exit ($\tilde{\mu} = \mu$)

$\mu \in (-1, 0)$: 0 is regular. Let $\tilde{\mu} = \mu$, then 0 is reflecting; Let $\tilde{\mu} = |\mu|$, then 0 is killing

$\mu \leq -1$: 0 is exit-not-entrance ($\tilde{\mu} = |\mu|$)

Reduction of the CIR model to the SQB process

SQB $dX_t = \lambda_0 dt + \nu\sqrt{X_t}dW_s$

$\Downarrow$ $Y_t = e^{-\lambda_1 t}X_{s(t)}$, where $s(t) = \begin{cases} t & \text{if } \lambda_1 = 0 \\ \frac{e^{\lambda_1 t}-1}{\lambda_1} & \text{if } \lambda_1 \neq 0 \end{cases}$

CIR $dY_t = (\lambda_0 - \lambda_1 Y_t)dt + \nu\sqrt{Y_t}dW_t$

PDF $p^{(CIR)}(t; x_0, x) = e^{\lambda_1 t}p^{(SQB)}(s(t); x_0, e^{\lambda_1 t}x)$

The FHT Distribution of the CIR diffusion process

- The FHT is

$$\tau_0^{(CIR)} \equiv \inf\{t : Y_t = 0\} \overset{d}{=} \inf\{t : X_{s(t)} = 0\} = s^{-1}(\tau_0^{(SQB)})$$

- The FHT PDF is $q^{(CIR)}(x_0; \tau) = e^{\lambda_1 t}q^{(SQB)}(x_0; s(\tau))$
- Since $\mathbb{P}(\tau_0^{(CIR)} < \infty) = \mathbb{P}(\tau_0^{(SQB)} < s(\infty))$, there are two cases:

$$\lambda_1 \geq 0 \Rightarrow s(\infty) = \infty \Rightarrow \mathbb{P}(\tau_0^{(CIR)} < \infty) = 1$$
$$\lambda_1 < 0 \Rightarrow s(\infty) = \frac{1}{|\lambda_1|} < \infty \Rightarrow \mathbb{P}(\tau_0^{(CIR)} < \infty) < 1$$

Simulation of the CIR diffusion process

- Consider the sequential sampling $y_0 \to y_1 \to y_2 \to \cdots \to y_n$, $y_i := Y_{t_i}$ for $0 = t_0 < t_1 < t_2 < \cdots < t_n$
- Map the time partition: $s_i = s(t_i)$, $i = 1, 2, \ldots, n$
- Sample the FHT $\tau = \dfrac{y_0}{2Y}$, where $Y \sim \mathrm{Gamma}(|\mu|, 1)$. If $\tau > s(\infty)$, the CIR process is not absorbed at zero
- Sample the path $x_0 \to x_1 \to \cdots \to x_n$, $x_i := X_{s_i}$, of the SQB process conditional on $X_0 = y_0$ and $X_\tau = 0$
- For all $i > 0$ set $y_i = e^{-\lambda_1 t_i} x_i$

The CEV Diffusion Model

- Squared Bessel Process: $dX_t = \lambda dt + 2\sqrt{X_t} dW_t$
- Apply the change of variables $X_t = F_t^{-2\beta}/(\delta^2\beta^2)$ where $\beta = \frac{1}{\lambda-2}$
- *Constant elasticity of variance* (CEV) diffusion model: $dF_t = \delta F_t^{\beta+1} dW_t$.
- The General Form of No-drift Transition PDF:

$$p_0(t; F_0, F_t) = \frac{F_t^{-2\beta-3/2} S_0^{1/2}}{\delta^2|\beta|t} \exp\left(-\frac{F_t^{-2\beta} + F_0^{-2\beta}}{2\delta^2\beta^2 t}\right) I_{\frac{1}{2|\beta|}}\left(\frac{F_t^{-\beta}F_0^{-\beta}}{\delta^2\beta^2 t}\right),$$

- By means of a scale and time change

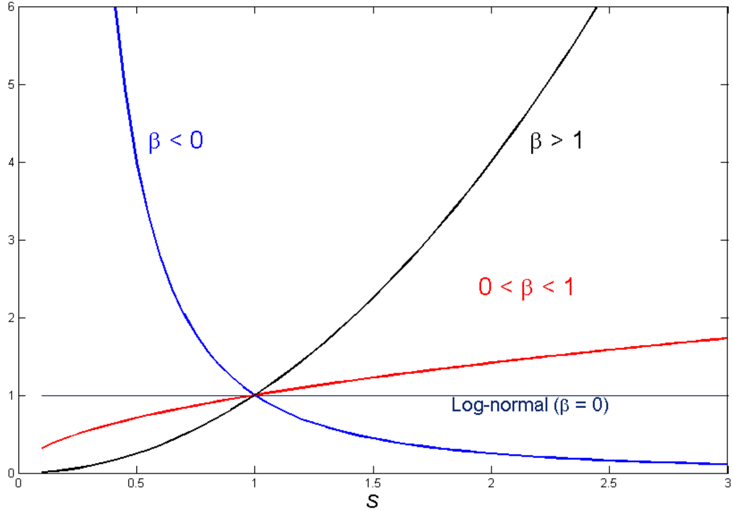
$$S_t = e^{rt} F_{s(t)}, \quad s(t) = \begin{cases} \frac{1}{2r\beta}\left(e^{2r\beta t} - 1\right), & r \neq 0, \\ t, & r = 0. \end{cases}$$

- The CEV Diffusion Process with Drift

$$dS_t = rS_t dt + \delta S_t^{\beta+1} d\tilde{W}_t$$

- The transition PDF is $p_r(t; S_0, S_t) = e^{-rt} p_0(\tau(t); S_0, e^{-rt} S_t)$

CEV Model – Local Volatility $\sigma(S)/S = \delta S^\beta$



Reduction of CEV to SQB or CIR

CEV diffusion with drift

$$dS_t = r\,dt + \delta S_t^{\beta+1} dW_t$$

$$Y_t = \frac{S_t^{-2\beta}}{\delta^2 \beta^2}$$

$$\lambda_0 = 2 + \tfrac{1}{\beta}$$

$$\lambda_1 = 2r\beta$$

CIR diffusion process

$$dY_t = (\lambda_0 - \lambda_1 t)dt + \nu\sqrt{Y_t}\,dW_t$$

$$S_t = e^{rt} F_{s(t)} \qquad s(t) = \frac{e^{2r\beta t} - 1}{2r\beta}$$

$$Y_t = e^{-\lambda_1 t} X_{s(t)} \qquad s(t) = \frac{e^{\lambda_1 t} - 1}{\lambda_1}$$

CEV diffusion without drift

$$dF_t = \delta F_t^{\beta+1} dW_t$$

$$X_t = \frac{F_t^{-2\beta}}{\delta^2 \beta^2}$$

$$\lambda_0 = 2 + \tfrac{1}{\beta}$$

SQB diffusion process

$$dX_t = \lambda_0 dt + \nu\sqrt{X_t}\,dW_t$$

Direct Simulation Algorithm

- Let us sample a skeleton of a price process (S_t) over a discretized partition, $0 = t_0 < t_1 < \cdots < t_N = T$.
- Simulation Scheme:
  1. Sample the FHT τ_0 from the gamma distribution for the CEV model.
  2. Sample a path of the corresponding underlying process (SQB or CIR) conditional on τ_0
  3. Apply a change of variables to obtain a sample path of the diffusion model
- The main advantage is that such a simulation scheme allows us to avoid sampling from complicated probability distributions

Occupation Time Simulation for the CEV Process

- Transform CEV $\to$ CIR
- Draw a skeleton $X_{t_1}, \ldots, X_{t_N}$ of the CIR process
- Transform CIR $\to$ ROU (the radial Ornstein-Uhlenbeck process)

$$\begin{array}{rl} \text{CIR} & dY_t = (\lambda_0 - \lambda_1 Y_t)dt + 2\sqrt{Y_t}dW_t \\ \Downarrow & X_t = \sqrt{Y_t} \\ \text{ROU} & dX_t = \left(\frac{2\nu+1}{2X_t} - \gamma X_t\right) dt + dW_t \end{array}$$

- Simulate approximately the occupation time $A_T^L(S) = A_T^\ell(X)$, where $\ell = \frac{L^{-\beta}}{\delta|\beta|}$
- Return $A_T^L(S)$ and the exact skeleton of the CEV process
- To speed up the algorithm, we may first sample the FHT $\tau_L(S) = \inf\{t \geq 0 \,:\, S_t = L\}$.
    - Let $S_0 > L$.
    - If $\tau_L(S) > T$, then $A_T^{L,-} = T$ and $A_T^{L,+} = T$.
    - If $\tau_L(S) \leq T$, then sample $A_{T-\tau_L}^{L,\pm}$ for (S_t) starting at $S_0 = L$.

Summary of Results

- Considered pricing of occupation time derivatives
- Obtained the CDF of the occupation time of a Brownian bridge
- Proposed am approximate scheme for sampling the occupation time of a solvable diffusion process by using the Brownian bridge interpolation
- Developed new methods of the exact simulation of Bessel-type diffusions
  - Sequential algorithms
  - Bridge algorithm conditional on the FHT at zero
  - Full bridge algorithm (not presented here)
- Applied to pricing path-dependent options and occupation time-derivatives under state-dependent volatility models
- References:

1. Makarov R. & Glew. D., Exact Simulation of Bessel Diffusions, submitted (see arXiv)
2. Campolieti, G. & Makarov, R.N., On Properties of Analytically Solvable Families of Local Volatility Diffusion Models, Mathematical Finance, to appear