Vector Monte Carlo estimators: dual presentations and optimization

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CONTENTS

$$\phi_i(x) = \sum_{j=1}^{m} \int_X k_{ij}(x,y)\phi_j(y)dy + h_i(x) \qquad (1)$$

or $\Phi = \mathbf{K}\Phi + H$, where $H^{\mathrm{T}} = (h_1, \ldots, h_m)$, $\mathbf{K} \in [L_\infty \to L_\infty]$,
$\Phi^{\mathrm{T}} = (\phi_1, \ldots, \phi_m)$, $||H||_{L_\infty} = \text{vrai} \sup_{i,x} |h_i(x)|$.

Integration with respect to Lebesgue measure in κ–dimensional Euclidean space X.

Let us suppose that spectral radius $\lambda(\mathbf{K}) < 1$ and

$$\Phi = \sum_{n=0}^{\infty} \mathbf{K}^n H. \qquad (2)$$

Since

$$\lambda(\mathbf{K}) = \underline{\lim} ||\mathbf{K}^n||^{1/n} = \inf ||\mathbf{K}^n||^{1/n},$$

then the Neumann series (2) is convergent if $||\mathbf{K}^{n_0}|| < 1$ for some $n_0 \geq 1$. Let us note that

$$||\mathbf{K}|| = \sup_{x,i} \sum_{j=1}^{m} \int |k_{ij}(x,y)|dy.$$

Let us consider Markov chain $\{x_n\}(n = 0, \dots, N)$, transition density $p(x, y)$ and $p(x) = 1 - \int_X p(x, y)dy \geq 0$ – probability of termination at the point x. Here N is the number of the state at which the trajectory terminates (i.e., the termination moment), $x_0 \equiv x$.

Vector collision estimator

$$\boldsymbol{\xi}_x = H(x) + \sum_{n=1}^{N} Q_n H(x_n),$$

$$Q_0 = I, \quad Q_{n+1} = Q_n K(x_n, x_{n+1})/p(x_n, x_{n+1}), \quad n = 0, 1, \dots,$$

where I– unity matrix, $K(x, y)$ – matrix with elements $\{k_{ij}(x, y)\}, (i, j = 1, \dots, m)$.

$$\boldsymbol{\xi}_x = H(x) + \sum_{n=1}^{\infty} \triangle_n Q_n H(x_n) \text{ and } \mathrm{E}(\triangle_n Q_n H(x_n)) = (\mathbf{K}^n H)(x)$$

under condition $p(x, y) > 0$, if $\sum_{i,j=1}^{m} |k_{ij}(x, y)| > 0$.

In accordance with $\Phi = \sum_{n=0}^{\infty} \mathbf{K}^n H$ and $\rho(\mathbf{K}_1) < 1$ we have $\mathrm{E}\boldsymbol{\xi}_x = \Phi(x)$.

Here $\mathbf{K}_1 = K_1(x, y)$ – matrix with elements $\{|k_{ij}(x, y)|\}, (i, j = 1, \dots, m)$.

Covariance matrix for vector collision estimator

In [7] there is the equation for covariance matrix $\Psi(x) = \mathrm{E}(\boldsymbol{\xi}_x \boldsymbol{\xi}_x^{\mathrm{T}})$:

$$\Psi(x) = \chi(x) + \int\limits_X \frac{K(x,y)\Psi(y)K^{\mathrm{T}}(x,y)}{p(x,y)} dy, \tag{3}$$

or $\Psi = \chi + \mathbf{K}_p\Psi$, where $\chi = H\Phi^{\mathrm{T}} + \Phi H^{\mathrm{T}} - HH^{\mathrm{T}}$ **on the assumption that** $\Psi \in \mathbf{L}_\infty$. The space $\mathbf{L}_\infty$ of matrix-value function with norm $||\Psi|| = \mathrm{vrai}\sup\limits_{i,j,x} |\Psi_{i,j}(x)|$.

$$\boldsymbol{\xi}_x = H(x) + \delta_x Q(x,y)\boldsymbol{\xi}_y, \tag{4}$$

where $Q(x,y) = K(x,y)/p(x,y)$ and δ_x – non-termination indicator in $x \to y$.

$$\boldsymbol{\xi}_x \boldsymbol{\xi}_x^{\mathrm{T}} = H(x)H^{\mathrm{T}}(x) + \delta_x q(x,y)\boldsymbol{\xi}_y H^{\mathrm{T}}(x) + \\ + \delta_x H(x)\boldsymbol{\xi}_y^T q^{\mathrm{T}}(x,y) + \delta_x q(x,y)\boldsymbol{\xi}_y \boldsymbol{\xi}_y^T q^{\mathrm{T}}(x,y) \tag{5}$$

Partial averaging (5) with respect to y and δ_x leads to (3).

But it is possible that $||\Psi|| = +\infty$ **even if** $\lambda(\mathbf{K}_p) < 1$

THEOREM 1 *If* $\lambda(\mathbf{K}_1) < 1$ *and* $\lambda(\mathbf{K}_{p,1}) < 1$ *then* $\Psi(x) = \mathrm{E}(\boldsymbol{\xi}_x \boldsymbol{\xi}_x^{\mathrm{T}})$ *is the solution of the* (3) *and* $\Psi \in \mathbf{L}_\infty$ *(* $\mathbf{K}_1 = K_1(x,y)$ *witn elements* $\{|k_{ij}(x,y)|\}, (i,j = 1,\ldots,m)$*)*
.

Dual presentation for the second moments of the vector collision estimators

$$I = (F, \Phi) = \int\limits_X F^{\mathrm{T}}(x)\Phi(x)dx,$$

where $F^{\mathrm{T}}(x) = (f_1(x), \dots, f_m(x))$ and $||F^{\mathrm{T}}||_{L_1} = \sum\limits_{j=1}^{m} \int\limits_X |f_j(x)|dx < \infty$. Let the point $x_0 \sim \pi(x)$ and $\pi(x) \neq 0$ if $F^{\mathrm{T}}(x)\Phi(x) \neq 0$. Then

$$I = \mathrm{E}\left\{\frac{F^{\mathrm{T}}(x_0)}{\pi(x_0)}\boldsymbol{\xi}_{x_0}\right\} = \mathrm{E}\left\{\sum_{n=0}^{N} \frac{F^{\mathrm{T}}(x_0)}{\pi(x_0)} Q_n H(x_n)\right\} = \mathrm{E}\left\{\sum_{n=0}^{N} H^{\mathrm{T}}(x_n) Q_n^{\mathrm{T}} \frac{F(x_0)}{\pi(x_0)}\right\}.$$

Random vector weights

$$Q_n^{(\mathrm{T})} = Q_n^{\mathrm{T}} F(x_0)\pi(x_0), \quad Q_n^{\mathrm{T}} = [K^{\mathrm{T}}(x_{n-1}, x_n)/p(x_{n-1}, x_n)]Q_{n-1}^{\mathrm{T}}.$$

$$\zeta = F^{\mathrm{T}}(x_0)\boldsymbol{\xi}_{x_0}/\pi(x_0) = \sum_{n=0}^{N} H^{\mathrm{T}}(x_n) Q_n^{(\mathrm{T})} F^{\mathrm{T}}/\pi(x_0), \tag{6}$$

$I = \mathrm{E}\zeta = (\Phi^*, H)$, where $\Phi^* = \mathbf{K}^*\Phi^* + F$ [2]

$$\mathrm{E}\zeta^2 = \mathrm{E}\{F^{\mathrm{T}}(x_0)\boldsymbol{\xi}_{x_0}\boldsymbol{\xi}_{x_0}^{\mathrm{T}}F(x_0)/\pi^2(x_0)\} = \mathrm{E}\{F^{\mathrm{T}}(x_0)\Psi(x_0)F(x_0)/\pi^2(x_0)\}. \qquad (7)$$

In particular $\mathrm{D}\zeta < +\infty$ if $\lambda(\mathbf{K}_{p,1}) < 1 (\Psi = \chi + \mathbf{K}_p\Psi)$ and $F^{\mathrm{T}}(x)/\pi(x) \in L_1$.

For dual presentation we insert $\Psi = \sum\limits_{n=0}^{\infty} \mathbf{K}_p\chi,\ \chi = H\Phi^{\mathrm{T}} + \Phi H^{\mathrm{T}} - HH^{\mathrm{T}}$ to (7).

$$\left[\frac{F^{\mathrm{T}}(x_0)}{\pi(x_0)}\frac{K(x_0,x_1)}{p(x_0,x_1)}\dots\frac{K(x_{n-1},x_n)}{p(x_{n-1},x_n)}H(x_n)\right]\times\left[H^{\mathrm{T}}(x_n)K^{\mathrm{T}}(x_{n-1},x_n)\dots K^{\mathrm{T}}(x_0,x_1)\frac{F(x_0)}{\pi(x_0)}\right]$$

$$= \left[H^{\mathrm{T}}(x_n)\frac{K^{\mathrm{T}}(x_{n-1},x_n)}{p(x_{n-1},x_n)}\dots\frac{K^{\mathrm{T}}(x_0,x_1)}{p(x_0,x_1)}\frac{F(x_0)}{\pi(x_0)}\right]\times\left[\frac{F^{\mathrm{T}}(x_0)}{\pi(x_0)}K(x_0,x_1)\dots K(x_{n-1},x_n)H(x$$

$$\mathrm{E}\zeta^2 = \int\limits_X \big(H^{\mathrm{T}}(x)\widetilde{\psi}(x)\Phi(x) + \Phi^{\mathrm{T}}(x)\widetilde{\psi}(x)H(x) - H^{\mathrm{T}}(x)\widetilde{\psi}(x)H(x)\big)dx, \qquad (8)$$

where $\widetilde{\psi}$ – Neumann series for

$$\widetilde{\psi}(x) = \int\limits_X \frac{K^{\mathrm{T}}(y,x)\widetilde{\psi}(y)K(y,x)}{p(y,x)}dy + \frac{F(x)F^{\mathrm{T}}(x)}{\pi(x)} \qquad (9)$$

or $\widetilde{\psi} = \widetilde{\mathbf{K}}_p\widetilde{\psi} + FF^{\mathrm{T}}/\pi$.

Here $\widetilde{\psi}$ – some covariation function. Therefore $\widetilde{\psi}_{ii}(x) \geq 0$, $|\widetilde{\psi}_{ij}(x)| \leq (\widetilde{\psi}_{ii}(x)\widetilde{\psi}_{jj}(x))^{1/2}$

$$\widetilde{\psi}_{ii}(x) \leq \int\limits_X \frac{\left(|K^{\mathrm{T}}(y,x)|\widetilde{\chi}(y)\right)^2_{ii}}{p(y,x)} dy + \frac{F_i^2(x)}{\pi(x)},$$

where $\widetilde{\chi}$ – vector with components $(\widetilde{\psi_{ii}})^{1/2}$.

$$\widetilde{\psi}_{ii}(x) \leq \int\limits_X \frac{\left(\sum\limits_{j=1}^m |k_{ji}(y,x)|(\widetilde{\psi}_{jj}(y))^{1/2}\right)^2}{p(y,x)} dy + \frac{F_i^2(x)}{\pi(x)} \leq \int\limits_X \frac{\sum\limits_{j=1}^m k_{ji}^2(y,x) \sum\limits_{j=1}^m \widetilde{\psi}_{jj}(y)}{p(y,x)} dy + \frac{F_i^2(x)}{\pi(x)}$$

Therefore (see [7], Lemma 1.1) we have

$$\overline{\widetilde{\psi}}(x) = \sum_{i=1}^m \widetilde{\psi}_{ii}(x) \leq g(x), \quad g(x) = \int\limits_X \frac{\bar{k}^2(y,x)g(y)}{p(y,x)} dy + \frac{\bar{f}^2(x)}{\pi(x)}, \qquad (10)$$

and

$$\bar{k}^2(y,x) = \sum_{i=1}^m \sum_{j=1}^m k_{ji}^2(y,x), \quad \bar{f}(x) = \sum_{i=1}^m F_i^2(x).$$

The majorant mean-square error minimization of the global solution estimator (histogram type)

$$\Phi^* = \mathbf{K}^*\Phi^* + F, \quad \Phi_i^*(x) = \sum_{j=1}^{m} \int\limits_X k_{ji}(y,x)\Phi_j^*(x)dx + F_i(x).$$

To construct an estimator for Φ^* of histogram type we use

$$H_r^{(i,s)}(x) = \begin{cases} 0, & r \neq i, \\ h_s(x) & r = i. \end{cases} \quad h_s(x) = \begin{cases} 1/\mathrm{mes}(D_s), & x \in D_s, \\ 0 & x \in X \backslash D_s. \end{cases} \tag{11}$$

for some space partition $\{D_s\}_{s=1,2,\ldots}$ of bounded $X \in \mathbb{R}^\kappa$. We have

$$(\Phi^*, H^{(i,s)}) = \int\limits_{D_s} \Phi_i^*(x)dx/\mathrm{mes}(D_s).$$

Vector collision estimator (6) we will define here as $\zeta^{(i,s)}$ and the solution of

$$\phi_i(x) = \sum_{j=1}^{m} \int\limits_X k_{ij}(x,y)\phi_j(y)dy + h_i(x)$$

as $\Phi^{(i,s)}$.

Let $\triangle = \max_s \text{mes } D_s$. From the expression (8) we obtain that if $||\widetilde{\psi}(x)|| < c < \infty$ we have $\mathrm{E}(\zeta^{(i,s)})^2 = O(\triangle^{-1})$ for $\triangle \to 0$ and $\mathrm{E}\zeta^{(i,s)} \equiv \Phi_i^*(x_s^{(i)}) \equiv O(1)$. Here $x_s^{(i)}$ – some point in D_s and we assume that $\Phi_i^*(x)$ are continuous in D_s, $i = 1, 2, ..., m, s = 1, 2, ...$.

$$\mathrm{D}\zeta^{(i,s)} \to \mathrm{E}(\zeta^{(i,s)})^2$$

If $K(y, x)$ sectionally continuous with respect to x in X for some bounded regular partition we have $\Phi_i(x_s) \asymp H_r^{(i,s)}(x)$ under $\triangle \to 0$ and

$$\mathrm{D}\zeta^{(i,s)} \sim (\widetilde{\psi}_{ii}, h_s^2) = \widetilde{\psi}_{ii}(x_s^{(i)})[\text{mes } D_s]^{-1}, \tag{12}$$

if $\widetilde{\psi}_{ii}(x)$ is continuous in D_s.

Let as suppose that X is partitioned by rectangular lattice and $\triangle = h^\kappa$.

$$|\Phi_i^*(x) - \mathrm{E}\widetilde{\Phi}_i^*(x)|^2 = c_i(x)h^2 + o(h^2), \quad c_i(x) < c < +\infty, \quad i = 1, 2, ..., m, \tag{13}$$

where $c = [\sup_{x,i} |\text{grad } \Phi_i^*(x)|]^2$ and $\widetilde{\Phi}^*(x)$ – approximate estimator of $\Phi(x)$ with the use of $\zeta^{(i,s)}$.

The problem of convergence $\widetilde{\Phi}^*$ to Φ^* we will study in some metric $B^2(\Phi^*, \widetilde{\Phi}^*) = \mathrm{E}||\Phi^* - \widetilde{\Phi}^*||^2_{L(X)}$ where $L(X)$ – proper Banach space because

$$P(||\Phi^* - \widetilde{\Phi}^*|| \geq \varepsilon) \leq \mathrm{E}||\Phi^* - \widetilde{\Phi}^*||^2/\varepsilon^2, \ \forall \varepsilon > 0.$$

$$\text{Let us define} \quad B^2(\Phi^*, \widetilde{\Phi}^*) = \sum_{i=1}^{m} \int\limits_X \rho^2(x) \mathrm{E}[\Phi_i^*(x) - \widetilde{\Phi}_i^*(x)]^2 dx =$$

$$= \sum_{i=1}^{m} \int\limits_X \rho^2(x) \mathrm{D}\widetilde{\Phi}_i^*(x) dx + \sum_{i=1}^{m} \int\limits_X \rho^2(x) [\Phi_i^*(x) - \mathrm{E}\widetilde{\Phi}_i^*(x)]^2 dx,$$

where $0 \leq \rho(x) \leq c_\rho < +\infty$. Under (12) and (13) and $h \to \infty$

$$B^2(\Phi^*, \widetilde{\Phi}^*) = O\left(C_v h^2 + \frac{d}{nh^\kappa}\right), \tag{14}$$

$$\text{where} \quad C_v = \sum_{i=1}^{m} \int\limits_X \rho^2(x) c_i(x) dx, \quad d = \sum_{i=1}^{m} \int\limits_X \rho^2(x) \widetilde{\psi}_{ii}(x) dx = \overline{\widetilde{\psi}\rho^2}$$

and n – amount of Markov chain sampling .

$$S = nt \to \min_{n,h}, \text{ under condition } C_v h^2 + \frac{d}{nh^\kappa} = \delta^2. \tag{15}$$

Here δ – demanded error and t – average cost of the statistical experiment

Solution $h^* = \left[\frac{\kappa}{(2+\kappa)C_v}\right]^{1/2}\delta$, $n^* = \left(d(2+\kappa)[(2+\kappa)C_v/\kappa]^{\kappa/2}\right)/2\delta^{2+\kappa}$ and

$$S^* = t\frac{d(2+\kappa)[(2+\kappa)C_v/\kappa]^{\kappa/2}}{2\delta^{2+\kappa}}. \tag{16}$$

Problem: minimize $d = \overline{\widetilde{\psi}\rho^2}$ or its majorant $\int_X g(x)\rho^2(x)dx = \overline{g\rho^2}$ by proper choice of the $p(y,x)$ and $\pi(x)$.

On the basis of the (10) we have

$$\overline{g\rho^2} = \int_X \left(\int_X \frac{\bar{k}^2(y,x)\rho^2(x)dx}{p(y,x)\rho^2(y)} \right) \rho^2(y)g(y)dy + \int_X \frac{\bar{f}^2(x)\rho^2(x)}{\pi(x)}dx. \tag{17}$$

An assumption is that $\overline{g\rho^2} \to min$ if $\overline{K}\rho \equiv q_v\rho$ (here ρ – main eigenfunction of the integral operator $\overline{K}$ with kernel $\bar{k}(y,x) = \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} k_{ji}^2(y,x)\Big)^{1/2}$) and

$$p_\rho(y,x) = \frac{\bar{k}(y,x)\rho(x)}{q_v\rho(x')}, \quad q_\rho(y) = \int_X \bar{k}(y,x)\rho(x)dx = [\overline{K}\rho](y) = q_v\rho(y). \tag{18}$$

THEOREM 2 *Let $\bar{k}(y,x)\rho(x)$ and $\bar{f}(x)\rho(x)$ are sectionally continuous and bounded for given finite partition of X,*

$$\rho(x) < c < \infty, \; \bar{k}(y,x)\rho(x) \geq \varepsilon > 0, \; q_v = \lambda(\overline{K}) < 1, \quad \textit{and} \;\; \rho \equiv q_v\overline{K}\rho.$$

$$\textit{Then for} \;\; p_\rho(y,x) \;\textit{and}\; \pi_\rho(x) = \bar{f}(x)\rho(x)/\int_X \bar{f}(x)\rho(x)dx \tag{19}$$

variable $\overline{g\rho^2}$ is minimum and

$$\overline{g\rho^2} = C_v(1-q_v^2)^{-1}, \quad C_v = \int_X \bar{f}^2(x)\rho^2(x)/\pi_\rho(x)dx = \Bigg(\int_X \bar{f}(x)\rho(x)dx\Bigg)^2.$$

Corollary *If $\rho \equiv 1$ and $\int_X \bar{k}(y,x)dx \equiv q_v$ then $p(y,x) = \bar{k}(y,x)q_v^{-1}$.*

Comparison between scalar and vector algorithms

Let is introduce additional discrete random parameter $i = 1, 2, \ldots, m$.
System (1) can be rewritten as one equation

$$\phi(i, x) = \sum_{j=1}^{m} \int_X k((i, x), (j, y))\phi(j, y)dy + h(i, x), \qquad (20)$$

or $\phi = K\phi + h$. Here $k((i, x), (j, y)) = k_{ij}(x, y), h(i, x) = h_i(x)$ and $\phi(i, x) = \phi_i(x)$.

Markov chain $(i_0, x_0), (i_1, x_1), \ldots, (i_N, x_N)$, Density distribution $p_{ij}(x, y) = p((i, x), (j$
for transition $(i, x) \to (j, y)$ in Markov chain is defined as

$$P(i \to j|x) = p_{ij}(x) = \int_X p_{ij}(x, y)dy, \quad \sum_{j=1}^{m} p_{ij}(x) = q_i(x) \le 1, \quad i, j = 1, \ldots, m.$$

Variable $p_i(x) = 1 - q_i(x)$ is termination trajectory probability under transition from (i, x). If the event $i \to j$ is true then the next transition probability is

$$r_{ij}(x, y) = p_{ij}(x, y)/q_i(x).$$

Scalar collision estimator

$$\xi_{(i,x)} = h(i,x) + \delta_{(i,x)} q((i,x),(j,y)) \xi_{(j,y)}, \tag{21}$$

where

$$q((i,x),(j,y)) = k((i,x),(j,y))/p((i,x),(j,y))$$

and $\delta_{(i,x)}$ – indicator of non-termination

$$P(\delta_{(i,x)} = 1) = q_i(x), \quad P(\delta_{(i,x)} = 0) = 1 - q_i(x).$$

Conditions of unbiasedness

$$p((i,x),(j,y)) \neq 0, \quad \text{если} \quad k((i,x),(j,y)) \neq 0 \ \ \forall (i,x),(j,y). \tag{22}$$

If all elements are non-negative then under (22) $\mathrm{E}\xi_{(i,x)} = \phi(i,x)$ is Neumann series for the (20), a $\psi(i,x) = \mathrm{E}\xi^2_{(i,x)}$ is Neumann series for the $\psi = K_p\psi + \chi$

$$\psi(i,x) = \sum_{j=1}^{m} \int\limits_X \frac{k^2((i,x),(j,y))}{p((i,x),(j,y))} \, \psi(j,y) dy + \chi(i,x),$$

where $\chi \equiv h(2\phi - h)$.

$$\text{Here for scalar estimator } \quad ||K_p||_{L_\infty} = \sup_{i,x} \sum_{j=1}^{m} \int_X \frac{k^2((i,x)(j,y))}{p((i,x)(j,y))} dy \tag{23}$$

Let us compare vector and scalar estimators

$$\text{If } \quad p((i,x)(j,y)) = m^{-1} p(x,y), \quad \int_X p(x,y)dy \le 1, \tag{24}$$

then
$$\sum_{j=1}^{m} \int_X \frac{k^2((i,x)(j,y))}{m^{-1}p(x,y)} dy = \int_X \frac{m \sum_{j=1}^{m} k^2((i,x)(j,y))}{p(x,y)} dy \ge \int_X \frac{\left(\sum_{j=1}^{m} |k_{ij}(x,y)| \right)^2}{p(x,y)} dy.$$

$$\text{For vector estimator } \quad ||\mathbf{K}_p||_{\mathbf{L}_\infty} = \sup_{i,x} \int \frac{\left(\sum_{j=1}^{m} |k_{ij}(x,y)| \right)^2}{p(x,y)} dy. \tag{25}$$

In the case of (24) we have $||K_p|| \ge ||\mathbf{K}_p||$.

On the other hand in the case of

$$k_{ij}(x, y) \geq 0 \quad \text{и} \quad q_i(x) \leq 1,\ i, j = 1, \ldots, m,$$

scalar algorithm allows to use the direct simulation ($p_{ij} \equiv k_{ij}$ and $q((i, x), (j, y)) \equiv 1$). In this case $K_p \equiv K$ e.g. $||K_p|| = ||K||$. Besides $||\boldsymbol{K}_p|| = ||K||$ if $p(x, y) = \sum_{j=1}^{m} k_{i_0 j}(x, y)$ where $i_0 = \arg\max_i q_i(x)$.

Let us note that vector algorithms are knowingly better than scalar algorithms for estimating the ratio or the difference of the solution vector components.

Problem: minimize

$$\overline{\widetilde{\psi}\rho^2} = \sum_{i=1}^{m} \int\limits_{D} \widetilde{\psi}(i,x)\rho_i^2(x)dx,$$

where

$$\widetilde{\psi}(i,x) = \sum_{j=1}^{m} \int\limits_{X} \frac{k^2((j,y),(i,x))}{p_{ji}(y,x)} \widetilde{\psi}(j,y)dy + \frac{f_i^2(x)}{\pi(i,x)}$$

by proper choice of the $p_{ji}(y,x)$ and $\pi(i,x)$.

THEOREM 3 *Let* $k_{ij}(x,y)\rho_i(y)$ *and* $f_i^2(x)\rho_i^2(x)$ *are sectionally continuous and bounded for given finite partition of* X*,*

$$\rho_i(x) < c < +\infty, \quad |k_{ij}(x,y)|\rho_j(y) \geq \varepsilon > 0, \quad q_s = \lambda(K_1) < 1 \ u \ \rho = q_s K_1 \rho.$$

Then for

$$p_\rho((j,y),(i,x)) = \frac{|k((j,y),(i,x))|\rho_i(y)}{q\rho_i(x)}$$

and

$$\pi_\rho(i,x) = |f_i(x)|\rho_i(x) / \sum_{i=1}^{m} \int_D |f_i(x)|\rho_i(x)dx$$

the variable $\overline{\widetilde{\psi}\rho^2}$ *is minimum and equal to*

$$(\overline{\widetilde{\psi}\rho^2})_s = C_s(1-q_s^2)^{-1}, \quad C_s = \left(\sum_{i=1}^{m} \int_D |f_i(x)|\rho_i(x)dx\right)^2.$$

Vector estimators for multiple parametric derivatives

Let us consider

$$\varphi(x,\sigma) = \int_X k(x,y,\sigma)\varphi(y,\sigma)dy + h(x,\sigma), \qquad (26)$$

or $\varphi = K\varphi + h$ in space $L_\infty(X)$.

Problem : estimation of the derivatives производных:

$$\varphi^{(n)}(x,\sigma) = \frac{\partial^n \varphi(x,\sigma)}{\partial \sigma^n}.$$

Let $K^{(n)}$ is integral operator with kernel $k^{(n)}(x,y,\sigma)$. By differentiating (26) m times with respect to σ we have triangular system of integral equations

$$\varphi^{(n)} = \sum_{i=0}^{n} C_n^i K^{(n-i)} \varphi^{(i)} + h^{(n)}, \qquad n = 0, 1, \ldots, m. \qquad (27)$$

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