Vladimír Baláž, Ladislav Mišík, Oto Strauch, János Tóth

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By *distribution function* (d.f.) we mean any nondecreasing function $g\colon [0,1] \to [0,1]$ with $g(0)=0$ and $g(1)=1$. Any two d.f. coinciding at all points of continuity are identified.

The set $\mathcal{D}$ of all distribution functions can be endowed with the metric ϱ defined by

$$\varrho(f,g) = \left(\int_0^1 (f(x) - g(x))^2 \, dx \right)^{\frac{1}{2}}, \quad f, g \in \mathcal{D}.$$

$(\mathcal{D}, \varrho)$ is a compact space and $\lim_{n\to\infty} g_n = g$ if and only if $\lim g_n(x) = g(x)$ holds at each point of continuity of g.

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Let $T = (t_1, t_2, \dots, t_N)$ be a finite sequence in $[0, 1)$. For $x \in [0, 1]$ denote

$$A([0, x); T) = \#\{n \le N;\ t_n < x\}$$

and define

$$F_T(x) = \frac{A([0, x); T)}{N}$$

the *step distribution function* of $T = (t_n)_{n=1}^{N}$.

Let $T_n = (t_{1n}, t_{2n}, \dots, t_{Nn}),\ n = 1, 2, \dots$ be an infinite sequence of finite sequences (blocks) in $[0, 1)$. The set

$$G(T_n) = \{\lim_{k\to\infty} F_{T_{n_k}};\ n_k \in \mathbb{N}\}$$

is called *the set of distribution functions of the block sequence* (T_n). It is always nonempty and closed.

Standard example: Let (x_n) be an infinite sequence in $[0, 1)$ and $X_n = \{x_1, \dots, x_n\}$. In this case $G(X_n)$ is denoted by $G(x_n)$. $G(x_n)$ is always connected.

Theorem. For each nonempty closed connected set $H \subset \mathcal{D}$ there exists (x_n) in $[0, 1)$ such that $G(x_n) = H$.

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Let $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$. The sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \dots, \frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called *ratio sequence of X* and $X_n = \{\frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}\}$ its *n-th block*.

Theory of $G(X_n)$ initiated by Strauch and Tóth ([ST] Publ. Math. Debrecen 2001), later by Grekos and Strauch ([GS] UDT 2007).

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There are disconnected sets $G(X_n)$.

On the other hand, there are nonempty closed connected sets $H \subset \mathcal{D}$ such that there is no $X \subset \mathbb{N}$ with $H = G(X_n)$. As an example of such set can serve

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[ST, Theorem 6.2, 6.3].

(A) If $\overline{d}(x_n) > 0$ then there exists $g \in G(X_n)$ such that $g(x) \le x$ for all $x \in [0, 1]$.

(B) If $\underline{d}(x_n) > 0$ then

- **(i)** there exists $g \in G(X_n)$ such that $g(x) \ge x$ for all $x \in [0, 1]$,
- **(ii)** for every $g \in G(X_n)$ we have

$$(\underline{d}(x_n)/\overline{d}(x_n))x \le g(x) \le (\overline{d}(x_n)/\underline{d}(x_n))x, \quad x \in [0, 1],$$

- **(iii)** every $g \in G(X_n)$ is continuous,

(C) If $d(x_n) > 0$ then $G(X_n) = \{x\}$.

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Theorem. [BMST] For every $X \subset \mathbb{N}$ (independently of $\underline{d}(X)$) there exists $g \in G(X_n)$ such that $g(x) \geq x$ for all $x \in [0,1]$.

The bounds in

$$(\underline{d}(x_n)/\overline{d}(x_n))x \leq g(x) \leq (\overline{d}(x_n)/\underline{d}(x_n))x, \quad x \in [0,1],$$

are symmetric. Is the situation really symmetric?

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Individual bounds Let $\underline{d}(X) > 0$ and $g \in G(X_n)$. Suppose that $\mathbf{n} = (n_k)$ is an increasing sequence of positive integers such that

$$g(x) = \lim_{k\to\infty} F(X_{n_k}, x)$$

and

$$d = d_{\mathbf{n}} = \lim_{k\to\infty} \frac{n_k}{x_{n_k}} \in [\underline{d}(X), \overline{d}(X)].$$

We call $d = d_{\mathbf{n}}$ a local density corresponding to **n.**
Then for all $0 \leq x < y \leq 1$ we have

$$0 \leq \frac{g(y) - g(x)}{y - x} \leq \frac{1}{d}. \tag{2}$$

Using (2) we can show that

$$h_{1,d}(x) \le g(x) \le h_{2,d}(x) \tag{3}$$

for all $x \in [0,1]$, where

$$h_{1,d}(x) = \begin{cases} \frac{\underline{d}}{d}\, x, & \text{if } x < x_d = \frac{1-d}{1-\underline{d}}; \\ \frac{1}{d}\, x + 1 - \frac{1}{d} & \text{if } x_d \le x \le 1, \end{cases} \tag{4}$$

$$h_{2,d}(x) = \min\left(\frac{\overline{d}}{d}\, x, 1\right). \tag{5}$$

Theorem. [BMST] Let $\underline{d}(X) > 0$. Then all d.f. in $G(X_n)$ are bounded by $h_1(x) \le g(x) \le h_2(x)$, where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise,} \end{cases} \tag{6}$$

$$h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right). \tag{7}$$

Question: Are these new bounds the best possible?

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Question: Are these new bounds the best possible?

Yes. Given any two numbers $0 < \alpha < \beta \leq 1$ there exists a set $X \subset \mathbb{N}$ with

$$\underline{d}(X) = \alpha, \qquad \overline{d}(X) = \beta$$

and such that

$$h_2 \in G(X_n),$$

$$\exists g \in G(X_n);\ g(x) = h_1(x)\ \forall x \in \left[0, \frac{1-\overline{d}(X)}{1-\underline{d}(X)}\right],$$

$$\forall x \in \left[\frac{1-\overline{d}(X)}{1-\underline{d}(X)}, 1\right] \exists f \in G(X_n);\ f(x) = h_1(x).$$

1. Introduction
2. Distribution functions and sets of positive integers
3. $G(X_n)$ and asymptotic densities of X
4. **Related results**

Theorem. [BMST] For every $X \subset \mathbb{N}$ with $\underline{d}(X) > 0$, the lower d.f. $\underline{g}(x) = \inf G(X_n)$ and the upper d.f. $\overline{g}(x) = \sup G(X_n)$ satisfy

$$\underline{g}(x).\underline{g}(y) \leq \underline{g}(x.y) \leq \overline{g}(x.y) \leq \overline{g}(x).\overline{g}(y) \tag{8}$$

for every $x, y \in (0, 1)$.

Inspiration: The functions $\underline{g} = h_1$ and $\overline{g} = h_2$ satisfy (8).

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Theorem. [BMST] For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with $0 < \underline{d}(x_n)$ we have

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}, \tag{9}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} \leq \tag{10}$$

$$\leq \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}},\overline{d})}{1-\underline{d}}\right)\left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}},\overline{d})}\right).$$

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Open problem. Characterize sets $H \subset \mathcal{D}$ for which there exists a set $X \subset \mathbb{N}$ such that $H = G(X_n)$.

Theorem. [BMST] Let H be a nonempty set of d.f.s defined on $[0, 1]$. Then there exists a set $X \subset \mathbb{N}$ such that $H \subset G(X_n)$.

Remark. It is sufficient to prove the theorem for $H = \mathcal{D}$. In this case necessarily $d(X) = 0$.

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In general, the smallest $G(X_n)$ with $H \subset G(X_n)$ need not exist.

Example. Let $H = \{c_0, c_1\}$, where $c_\alpha = \chi_{(\alpha,1]}$. Then there is no $X \subset \mathbb{N}$ such that $H = G(X_n)$. On the other hand, one can find $Y \subset \mathbb{N}$ such that

$$G(Y_n) = \{\alpha\chi_{(0,1)};\ \alpha \in [0,1]\}$$

and $Z \subset \mathbb{N}$ such that $G(Z_n) \cap G(Y_n) = H$.

Remark. In the above example $G(Y_n)$ is minimal, i.e. there is no $T \subset \mathbb{N}$ with $H \subset G(T_n) \subset G(Y_n)$.

Problem. Is it true that for every $H \subset \mathcal{D}$ there exists minimal $G(X_n)$ with $H \subset G(X_n)$?

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