Deriving Numerical Methods for SDEs driven by Fractional Brownian Motions

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SDEs driven by FBMs

$$(SDE) \qquad dX_t = b(X_t)dt + \sum_{j=1}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \qquad t \in [0,1]$$
$$X_0 = x_0 \in \mathbb{R}^d$$

where

- $b, \sigma^{(1)}, \ldots, \sigma^{(m)} : \mathbb{R}^d \to \mathbb{R}^d$
- $B^{(1)}, \ldots, B^{(m)}$ independent **fractional Brownian motions** with Hurst parameter $H \in (0,1)$, i.e. $B^{(j)}$ zero mean Gaussian process with continuous sample paths, $B_0^{(j)} = 0$ and

$$\mathbf{E}|B_t^{(j)} - B_s^{(j)}|^2 = |t-s|^{2H}, \qquad s,t \in [0,1]$$

SDEs driven by FBMs (cont'd)

$$(SDE) \qquad dX_t = b(X_t)dt + \sum_{j=1}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \qquad t \in [0,1]$$
$$X_0 = x_0 \in \mathbb{R}^d$$

Coutin, Qian (2002): Pathwise existence and uniqueness via rough paths theory for smooth b, $\sigma^{(j)}$ and $H > 1/3$

Lyons (1994), Lin (1995), Lyons (1998), Klingenhöfer, Zähle (1999); Nualart, Răşcanu (2002); Errami, Russo (2003); Gubinelli (2004); Friz, Victoir (2010); ...

$H = 1/2$: Stratonovich SDE driven by Brownian motion
$H > 1/2$: Riemann-Stieltjes (resp. Young) integral equation

Other Approach: Wick-Itô integral, see talk of `Peter Parczewski`

SDEs driven by FBMs (cont'd)

$$\sigma = (\sigma^{(1)}, \ldots, \sigma^{(m)}), \quad B = (B^{(1)}, \ldots, B^{(m)})$$

$$(SDE) \quad dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \in [0,1], \qquad X_0 = x_0 \in \mathbb{R}^d$$

Pathwise existence and uniqueness for smooth b, σ and $H > 1/3$, in particular:

$$X = F(x_0, B, \mathbf{B}^2)$$

where

- F locally Lipschitz in appropriate Hölder spaces
- $\mathbf{B}^2$ Lévy area associated to B, i.e.

$$\mathbf{B}^2_{st}(i,j) := \int_s^t (B_u^{(i)} - B_s^{(i)}) \, d^\circ B_u^{(j)}, \; i,j = 1, \ldots, m, \; 0 \leq s \leq t \leq 1$$

(d°: symmetric Russo-Vallois integral)

Numerical Methods: The Problem

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \in [0,1], \qquad X_0 = x_0 \in \mathbb{R}^d$$

Numerical problem: Approximation of X on $[0,1]$

Here: Construction of approximation Z^n based on

$$(\star) \quad \begin{cases} \text{(i)} & B_{1/n}, B_{2/n}, \ldots, B_1, \\ \text{(ii)} & x_0 \text{ and evaluations of } b, \sigma \text{ and derivatives,} \end{cases}$$

i.e. Z^n **implementable** numerical scheme

Remark

Exact simulation of $B_{1/n}, B_{2/n}, \ldots, B_1$ with cost $\mathcal{O}(n\log(n))$ by method of Wood-Chan and Davies-Harte

Assumptions

$b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$ bounded, $\sigma \in C_b^3(\mathbb{R}^d;\mathbb{R}^{d,m})$ bounded, $H > 1/3$

Known Results

Euler scheme

- Pathwise $\|\cdot\|_\infty$-convergence rate $n^{-2H+1+\varepsilon}$ for all $\varepsilon > 0$

  Davie (2007); Mishura, Shevchenko (2008)
- $m = d = 1$: pathwise $\|\cdot\|_\infty$-convergence rate n^{-2H+1}

  N, Nourdin (2007)

Milstein scheme

- Pathwise $\|\cdot\|_\infty$-convergence rate $n^{-\min\{3H-1,H\}+\varepsilon}$ f.a. $\varepsilon > 0$

  Davie (2007); Friz, Victoir (2010)
- Method uses $\mathbf{B^2}$: not implementable, since distribution of $\mathbf{B}^{\mathbf{2}}_{st}(i,j)$ unknown for $i \neq j$

Remark

$H = 1/2$: Euler scheme converges to Itô solution

Modified Milstein Scheme

Notation: $\Delta = \frac{1}{n}, \quad \Delta_k B = B_{(k+1)/n} - B_{k/n}$

$$
\begin{aligned}
Z_0^n &= x_0 \\
Z_{k+1}^n &= Z_k^n + b(Z_k^n)\Delta + \sigma(Z_k^n)\Delta_k B \\
&\qquad + \frac{1}{2}\Delta_k B\,(\sigma \cdot \partial\sigma)(Z_k^n)\,\Delta_k B \\
&\qquad + \frac{1}{2}\Delta\,(b \cdot \partial\sigma)(Z_k^n)\,\Delta_k B + \frac{1}{2}\Delta_k B\,(\sigma \cdot \partial b)(Z_k^n)\,\Delta
\end{aligned}
$$

where $(f \cdot \partial g)_{ij} = \sum_{q=1}^{d} f_q^{(i)} \partial_q g^{(j)}$

Extension to $[0,1]$ by piecewise linear interpolation

$$
Z_t^n = Z_k^n + (nt - k)(Z_{k+1}^n - Z_k^n), \qquad t \in [k/n, (k+1)/n)
$$

Now: derivation and convergence analysis

Step 1: Wong-Zakai Approximation

Replace B in (SDE) by

$$B_t^n = B_{k/n} + (nt - k)(B_{(k+1)/n} - B_{k/n}), \qquad t \in [k/n, (k+1)/n)$$

i.e. piecewise linear interpolation of B with stepsize $1/n$

$$(WZ) \qquad Y_t^n = x_0 + \int_0^t b(Y_s^n)ds + \int_0^t \sigma(Y_s^n)dB_s^n, \qquad t \in [0,1]$$

Then: (WZ) pathwise ordinary differential equation

$$\dot{Y}_t^n = b(Y_t^n) + \sigma(Y_t^n)\dot{B}_t^n, \quad t \in [0,1], \qquad Y_0^n = x_0$$

with

$$\dot{B}_t^n = n\Delta_k B, \qquad t \in (k/n, (k+1)/n)$$

Wong-Zakai Approximation

Notation: $\|f\|_{\kappa,\infty} = \sup_{t\in[0,1]} |f(t)| + \sup_{s,t\in[0,1]} \frac{|f(t)-f(s)|}{|t-s|^{\kappa}}$

Theorem I Deya, N, Tindel (2010)

$$\|X - Y^n\|_{\kappa,\infty} \leq \xi_{b,\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

for all $1/3 < \kappa < H$, where $\xi_{b,\sigma,\kappa,H}$ a.s. finite random variable

Proof Use universality and Lipschitz continuity of F:

$$\|X - Y^n\|_{\kappa,\infty} \leq \eta_{b,\sigma,\kappa,H} \cdot \left(\|B - B^n\|_{\kappa,\infty} + \sup_{s,t\in[0,1]} \frac{|\mathbf{B}^{\mathbf{2}}_{st} - \mathbf{B}^{\mathbf{2},n}_{st}|}{|s-t|^{2\kappa}} \right)$$

where $\mathbf{B}^{\mathbf{2},n}_{st}(i,j) = \int_s^t (B_u^{(i),n} - B_s^{(i),n})\, dB_s^{(j),n}$

Remark (WZ) semidiscretisation, in general not implementable

Step 2: Discretising the WZ-Approximation

(WZ) on $(k/n, (k+1)/n)$:

$$\dot{Y}_t^n = g_{n\Delta_k B}(Y_t^n)$$

with

$$g_\lambda(x) = b(x) + \sigma(x)\lambda, \qquad \lambda \in \mathbb{R}^m,\, x \in \mathbb{R}^d,$$

i.e.

$$\dot{Y}_t^n = b(Y_t^n) + n\sigma(Y_t^n)\Delta_k B$$

Discretise with 2nd order ODE Taylor scheme with stepsize $1/n$:

$$\begin{aligned} Z_{k+1}^n = Z_k^n &+ b(Z_k^n)\Delta + \sigma(Z_k^n)\Delta_k B \\ &+ \frac{1}{2}\Delta_k B\,(\sigma\cdot\partial\sigma)(Z_k^n)\,\Delta_k B \\ &+ \frac{1}{2}\Delta\,(b\cdot\partial\sigma)(Z_k^n)\,\Delta_k B + \frac{1}{2}\Delta_k B\,(\sigma\cdot\partial b)(Z_k^n)\,\Delta \end{aligned}$$

Modified Milstein Scheme

Discretising the WZ-Approximation

Theorem II Deya, N, Tindel (2010)

$$\|X - Z^n\|_{\kappa,\infty} \leq \xi_{b,\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

for all $1/3 < \kappa < H$

Remarks

- For b, σ unbounded: localisation
- Discretisation of (WZ) with 1st order scheme (e.g. Euler): no convergence to X
- Error bound sharp for $dY_t = dB_t$

Proof Theorem I and bounds for $Y^n - Z^n$:
One-step error $\approx n^{-3H+\varepsilon}$, control of error propagation via F

Discretising the WZ-Approximation: Extensions

(WZ) on $(k/n, (k+1)/n)$:

$$\dot{Y}_t^n = g_{n\Delta_k B}(Y_t^n)$$

with

$$g_\lambda(x) = b(x) + \sigma(x)\lambda, \qquad \lambda \in \mathbb{R}^m,\ x \in \mathbb{R}^d$$

Theorem II remains valid, if (WZ) is discretised by ODE method with numerical flow

$$\Psi_{g_\lambda}(x,h) = x + g_\lambda(x)h + (g_\lambda \cdot \partial g_\lambda)(x)\frac{h^2}{2} + (g_\lambda \cdot \partial(g_\lambda \cdot \partial g_\lambda))(\xi_{x,\lambda,h})\frac{h^3}{6}, \qquad x \in \mathbb{R}^d, h \in [0, h_*]$$

for some $\xi_{x,\lambda,h} \in \mathbb{R}^d$, where $h_* > 0$ independent of λ

Thus: Use of Heun's method, Runge-Kutta 4, second order drift implicit methods, ... possible ! (N, Rößler (2010))

Numerical Examples

Test for convergence rates of mod. Milstein scheme in $\| \cdot \|_\infty$-norm

Set

$$e(n) = \max_{k=0,\dots,n} |X_{k/n} - Z^n_{k/n}|$$

Smoothness of X:

$$\|X - Z^n\|_\infty \approx \sqrt{\log(n)} n^{-H} + e(n)$$

Conjecture:

$$e(n) \approx \sqrt{\log(n)} n^{-2H+1/2}$$

and

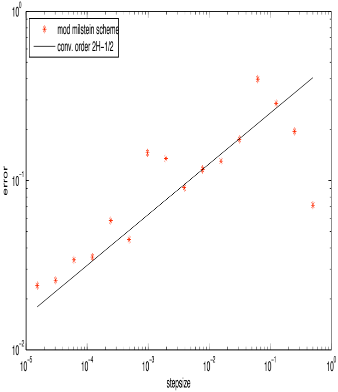
$$\|X - Z^n\|_\infty \approx \sqrt{\log(n)} \big(n^{-H} + n^{-2H+1/2}\big)$$

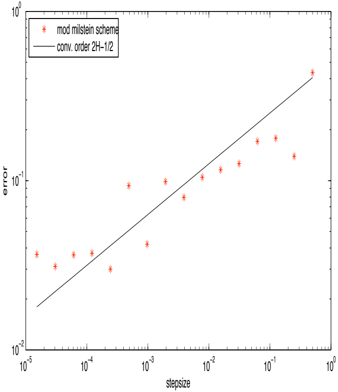
Based on error bounds for Lévy area; N, Tindel, Unterberger (2010)

Numerical Example I

$$dX_t = \cos(X_t)dB_t^{(1)} + \sin(X_t)dB_t^{(2)}, \quad X_0 = 1$$

with $H = 0.4$



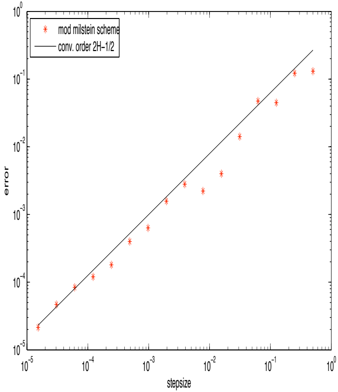


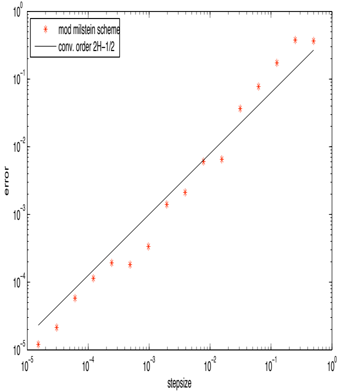
maximum error in discretization points

Numerical Example II

$$dX_t = \cos(X_t)dB_t^{(1)} + \sin(X_t)dB_t^{(2)}, \quad X_0 = 1$$

with $H = 0.7$



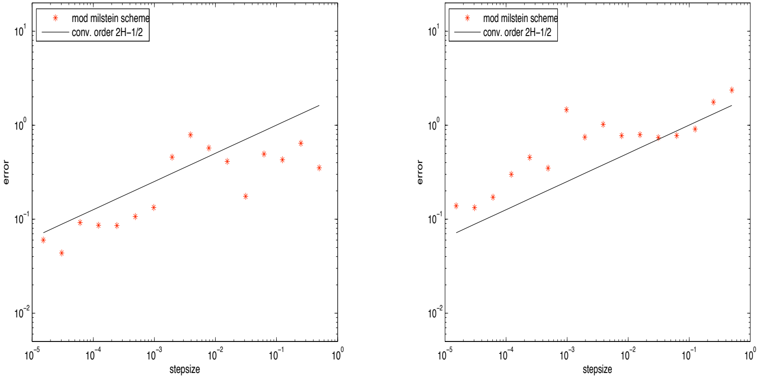


maximum error in discretization points

Numerical Example III

$$dX_t^{(1)} = Y_t^{(2)} dB_t^{(1)}, \quad dX_t^{(2)} = Y_t^{(1)} dB_t^{(2)}, \quad X_0^{(1)} = X_0^{(2)} = 2$$

with $H = 0.4$

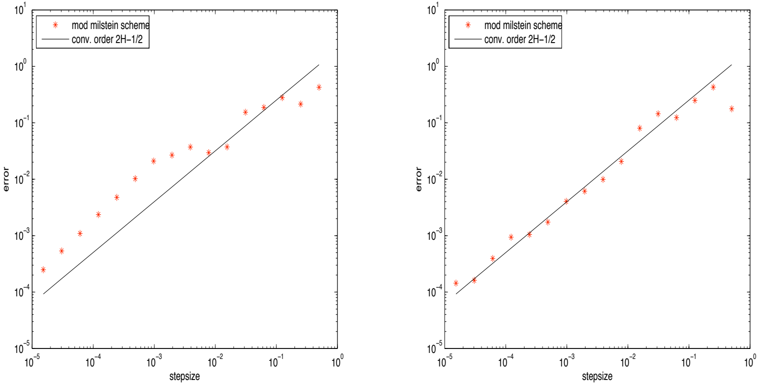


maximum error in discretization points

Numerical Example IV

$$dX_t^{(1)} = Y_t^{(2)} dB_t^{(1)}, \quad dX_t^{(2)} = Y_t^{(1)} dB_t^{(2)}, \quad X_0^{(1)} = X_0^{(2)} = 2$$

with $H = 0.7$



maximum error in discretization points

Summary

SDEs driven by FBMs with $H > 1/3$

- Definition, existence and uniqueness via rough paths theory

- Construction of convergent and implementable schemes:

  Discretise the Wong-Zakai approximation with ODE method of (at least) 2nd order

  Convergence rate in $\| \cdot \|_{\kappa,\infty}$-norm: $\sqrt{\log(n)}\, n^{-(H-\kappa)}$
  (for $1/3 < \kappa < H$)

Further result:
Extension to different Hurst parameters $H_j > 1/3$: N, Rößler, 2010