# The Discrepancy of Hybrid Sequences

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## MC vs. QMC methods

Monte Carlo (MC) methods are numerical methods based on sampling with random or pseudorandom numbers. Their typical convergence rate is $O(N^{-1/2})$, where $N$ is the sample size. Being stochastic methods, they allow statistical error estimation.

Quasi-Monte Carlo (QMC) methods are deterministic versions of MC methods. They replace sequences of pseudorandom numbers by low-discrepancy sequences. Their typical convergence rate is $O\left(N^{-1}\right)$ up to logarithmic factors.

## Hybrid sequences

An idea going back to Spanier (1995) is to combine the advantages of MC and QMC methods, i.e., statistical error estimation and faster convergence.

Such combined methods use hybrid sequences that are obtained by "mixing" different types of sequences, i.e., certain coordinates of the points stem from one type of sequence and the remaining coordinates from another type.

Of greatest practical interest is the combination of low-discrepancy sequences and sequences of pseudorandom numbers.

In view of the Koksma-Hlawka inequality, we need upper bounds on the discrepancy of hybrid sequences. Previous results by Ökten (1996), Ökten-Tuffin-Burago (2006), and Gnewuch (2009) provide only probabilistic bounds on this discrepancy. I describe the first deterministic bounds.

# The basic sequences

Low-discrepancy sequences:

(i) Halton sequences $\mathbf{y}_n = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n))$

(ii) Kronecker sequences $\mathbf{y}_n = \{n\boldsymbol{\alpha}\}$

Sequences of pseudorandom numbers:

(i) linear congruential sequences

(ii) nonlinear congruential sequences

(iii) explicit inversive sequences

(iv) digital inversive sequences

# Deterministic discrepancy bounds

$[0,1)^m$ $m$-dim. half-open unit cube ($m \geq 1$ arbitrary),

$\lambda_m$ $m$-dim. Lebesgue measure.

$J$ subinterval of $[0,1)^m$, $c_J$ characteristic function of $J$.

For $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1} \in [0,1)^m$ put

$$A(J;N) := \sum_{n=0}^{N-1} c_J(\mathbf{x}_n).$$

Discrepancy

$$D_N = \sup_J \left| \frac{A(J;N)}{N} - \lambda_m(J) \right| .$$

We consider hybrid sequences obtained by “mixing” a low-discrepancy sequence and a sequence of pseudorandom numbers (or vectors) from the two given lists. For many of the possible combinations, we have nontrivial deterministic discrepancy bounds. So far there are four papers on this topic by the speaker, one of them joint with Winterhof, starting from 2009.

In the following, we present a selection of the results.

## Halton + nonlinear congruential

Let $\mathbf{y}_0, \mathbf{y}_1, \ldots$ be an $s$-dim. Halton sequence with pairwise coprime bases $b_1, \ldots, b_s \geq 2$.

Let $p \geq 3$ be a prime with $\gcd(b_i, p) = 1$ for $1 \leq i \leq s$. Choose $g_1, \ldots, g_t \in \mathbb{F}_p[X]$ of distinct degrees with $2 \leq \deg(g_j) < p$ for $1 \leq j \leq t$. Then define the hybrid sequence

$$\mathbf{x}_n = \left(\mathbf{y}_n, \frac{g_1(n)}{p}, \ldots, \frac{g_t(n)}{p}\right) \in [0,1)^{s+t}, \ n = 0, 1, \ldots.$$

Let $D_N$ be the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}$.

**Theorem 1.**

$$D_N = O\left(\left(N^{-1} G_p \, p^{1/2} (\log p)^{t+1}\right)^{1/(s+1)}\right)$$

for $1 \leq N \leq p$, where $G_p = \max_{1 \leq j \leq t} \deg(g_j)$ and the implied constant depends only on $b_1, \ldots, b_s, t$.

**Remark 1.** Fix $b_1, \ldots, b_s, t$. For each prime $p \geq 3$, choose $N_p \in \mathbb{N}$ with $N_p \leq p$ such that

$$\lim_{p \to \infty} \frac{G_p \, p^{1/2} (\log p)^{t+1}}{N_p} = 0.$$

Then $D_{N_p} \to 0$ as $p \to \infty$ by Theorem 1, so by Koksma-Hlawka the corresponding hybrid MC method converges.

## Kronecker + nonlinear congruential

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ of type 1, e.g., the $\alpha_i$ are algebraic numbers such that $1, \alpha_1, \dots, \alpha_s$ are lin. indep. over $\mathbb{Q}$.

For a prime $p \geq 5$, choose $g_1, \dots, g_t \in \mathbb{F}_p[X]$ of distinct degrees with $3 \leq \deg(g_j) < p$ for $1 \leq j \leq t$. Then define the hybrid sequence

$$\mathbf{x}_n = \left( \{n\boldsymbol{\alpha}\}, \frac{g_1(n)}{p}, \dots, \frac{g_t(n)}{p} \right) \in [0,1)^{s+t}, \ n = 0, 1, \dots.$$

Let $D_N$ be the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$.

**Theorem 2.**

$$D_N = O\left(N^{-1/2}G_p^{1/2}p^{1/4}(\log p)^{1/2}(\log N)^{s+t}\right)$$

for $2 \leq N \leq p$, where $G_p = \max_{1\leq j\leq t} \deg(g_j)$ and the implied constant depends only on $\boldsymbol{\alpha}$ and $t$.

**Remark 2.** Fix $\boldsymbol{\alpha} \in \mathbb{R}^s$ of type 1 and $t \in \mathbb{N}$. For each prime $p \geq 5$, choose $N_p \in \mathbb{N}$ with $N_p \leq p$ such that

$$\lim_{p\to\infty} \frac{G_p\, p^{1/2}(\log p)(\log N_p)^{2(s+t)}}{N_p} = 0.$$

Then $D_{N_p} \to 0$ as $p \to \infty$ by Theorem 2, so by Koksma-Hlawka the corresponding hybrid MC method converges.

## Kronecker + explicit inversive

Let $\boldsymbol{\alpha} \in \mathbb{R}^s$ of type 1. For a prime $p \geq 5$, choose $a_1, \ldots, a_t \in \mathbb{F}_p^*$ and $d_1, \ldots, d_t \in \mathbb{F}_p$ such that $d_1/a_1, \ldots,$ $d_t/a_t$ are distinct elements of $\mathbb{F}_p$. For $1 \leq j \leq t$ and $n = 0, 1, \ldots,$ put

$$e_n^{(j)} = (a_j n + d_j)^{p-2} \in \mathbb{F}_p.$$

Then define the hybrid sequence

$$\mathbf{x}_n = \left( \{n\boldsymbol{\alpha}\}, \frac{e_n^{(1)}}{p}, \ldots, \frac{e_n^{(t)}}{p} \right) \in [0,1)^{s+t},\ n = 0, 1, \ldots .$$

**Theorem 3.** Let $D_N$ be the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \ldots,$ $\mathbf{x}_{N-1}$. Then

$$D_N = O\left(N^{-1/2} p^{1/4} (\log p)^{1/2} (\log N)^{s+t}\right)$$

for $2 \leq N \leq p$, where the implied constant depends only on $\boldsymbol{\alpha}$ and $t$.

Remarks 1 and 2 apply here in an analogous way.

## Kronecker + digital inversive

Let $\mathbb{F}_q$ be the finite field of order $q = p^k$, $p$ prime, $k \geq 2$. Let $\gamma_0, \gamma_1, \ldots \in \mathbb{F}_q$ be an inversive generator in the sense of H.N.-Rivat (2008) with least period $q+1$. For an ordered basis $\{\beta_1, \ldots, \beta_k\}$ of $\mathbb{F}_q$ over $\mathbb{F}_p$, we can write $\gamma_n = \sum_{l=1}^{k} c_{n,l}\, \beta_l$ with all $c_{n,l} \in \mathbb{F}_p$ being unique. Then a digital inversive sequence is defined by

$$z_n = \sum_{l=1}^{k} c_{n,l}\, p^{-l} \in [0,1) \qquad \text{for } n = 0, 1, \ldots .$$

Choose $\boldsymbol{\alpha} \in \mathbb{R}^s$ of type 1 and integers $0 \leq i_1 < i_2 < \cdots < i_t \leq q$. Then define the hybrid sequence

$$\mathbf{x}_n = (\{n\boldsymbol{\alpha}\}, z_{n+i_1}, \ldots, z_{n+i_t}) \in [0,1)^{s+t}$$

for $n = 0, 1, \ldots$ .

**Theorem 4.** Let $D_N$ be the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \ldots,$ $\mathbf{x}_{N-1}$. Then

$$D_N = O\left(N^{-1/4} q^{1/8} (\log q)^t (\log N)^s\right)$$

for $2 \leq N \leq q+1$, where the implied constant depends only on $\boldsymbol{\alpha}$ and $t$.

## The tools

The discrepancy bounds are proved by applying a version of the Erdös-Turán-Koksma inequality. In some cases the classical ETK inequality suffices. In other cases, where one constituent of the hybrid sequence is a digital sequence, a new version of the ETK inequality has to be used.

The remaining steps require bounds for exponential sums and (when Halton sequences are involved) counting arguments and elementary number theory.