Coin Tossing Algorithms for Integral Equations and Tractability

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See our book “Tractability of Multivariate Problems II” with H. Woźniakowski; joint work with S. Heinrich and H. Pfeiffer

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The Problem

Compute $u(s)$, integral equation

$$u(x) - \int_{[0,1]^d} k(x,y)u(y)\,\mathrm{d}y = f(x)$$

on $[0,1]^d$ with Lipschitz kernel k, $\|k\|_\infty < \alpha < 1$ and right hand side.

Deterministic algorithms: Optimal order $e_n \asymp n^{-1/(2d)}$, curse of dimension.

Optimal order with MC (Heinrich & Mathé 1993)

$$e_n \asymp n^{-1/2-1/(2d)}.$$

How much randomness is needed?
Can we work with "few" random bits?
Compare with recent book of Sugita.

Reduction of Randomness

In order to approximate $I_d(f) = \int_{[0,\,1]^d} f(x)\,dx$ up to error $\varepsilon > 0$ with the classical MC-method, one needs roughly ε^{-2} function values and

$$d \cdot \varepsilon^{-2} \quad \text{random numbers from } [0,1].$$

Trick of Bakhvalov (1964): Pairwise independent sample points for MC are ok, take

$$x_k = x + k \cdot y \qquad \text{mod } 1$$

with $2d$ random numbers for $x,\, y \in [0,1]^d$.

Hence $2d$ random numbers are enough.

What about random bits?

Approximation of Means

Approximate, for $f = (f(0), \ldots, f(N-1)) \in \mathbb{R}^N$,

$$S_N(f) = \frac{1}{N} \sum_{i=0}^{N-1} f(i).$$

Assume that f is from the set $F = \mathcal{B}(L_p^N)$, the unit ball of the space $L_p^N = L_p(\mu)$ where μ is the equidistribution on $\{0, 1, \ldots, N-1\}$ and $1 \le p \le \infty$.

$N = 2^s$: class. MC-meth. $A_n^\omega(f) = \frac{1}{n} \sum_{j=1}^n f(i_j^\omega)$

To implement this algorithm, we need $n \log_2 N$ random Bits.

Can we do better?

Approximation of Means

Bakhvalov's idea $\;\&\;$ field with N elements
obtain algorithm with the same error, n function values and $2\log_2 N$ random bits.

Illustration: choice of the i_j^ω for N prime:
Take $x, y \in \{0, 1, \ldots, N-1\}$ independently acc. to the uniform distribution and put

$$i_j^\omega = x + (j-1) \cdot y \quad \text{mod } N\,.$$

Heinrich, N., Pfeiffer (2004):
Let $1 \le p \le \infty$, $N \in \mathbf{N}$ and $\beta > 1$. Then, for $\beta n < N$,
$e_n(S_N, \mathcal{B}(L_p^N)) \asymp n^{-1/2}$ for $2 \le p \le \infty$ and
$e_n(S_N, \mathcal{B}(L_p^N)) \asymp n^{-1+1/p}$ for $1 \le p < 2$.
Roughly $\log_2 N$ random bits are needed.

Approximation of Integrals

Approximate

$$I_d(f) = \int_{[0,1]^d} f(x)\, dx$$

where f is from a *Sobolev class*:
$r \in \mathbf{N},\ 1 \le p \le \infty,\ D = [0,1]^d$

$$W_p^r(D) = \{f \in L_p(D) : \partial^\alpha f \in L_p(D), |\alpha| \le r\}$$

$$\|f\|_{W_p^r(D)} = \Big(\sum_{|\alpha| \le r} \|\partial^\alpha f\|_{L_p(D)}^p \Big)^{1/p}$$

∂^α weak partial derivative

$\mathcal{B}(W_p^r(D))$ unit ball of $W_p^r(D)$

Assume that $r/d > 1/p$ (Sobolev embedding condition).

Result for Algorithms that use Random Bits

Heinrich, N., Pfeiffer (2004):

Let $1 \leq p \leq \infty$, $r, d \in \mathbf{N}$, $r/d > 1/p$. Then there is a constant $c > 0$ such that the optimal order

$$e_n(I_d, \mathcal{B}(W_p^r(D))) \asymp n^{-r/d-1/2} \quad (2 \leq p \leq \infty)$$

$$e_n(I_d, \mathcal{B}(W_p^r(D))) \asymp n^{-r/d-1+1/p} \quad (1 \leq p < 2)$$

can be obtained with $c \log n$ random bits.

The upper bound is proven by a discretization technique: splitting the problem into a sequence of discrete summation problems.

The Integral Equation

Compute $u(s)$, integral equation

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on $[0,1]^d$ with Lipschitz kernel k, $\|k\|_\infty < \alpha < 1$ and right hand side.

Use the Neumann series and apply the above results for summation and/or integration and obtain, see N. & Pfeiffer (2004), the upper bound:

$$\text{cost} \leq \varepsilon^{-2} + d\,(\log \varepsilon^{-1})^2.$$

Need ε^{-2} function values and $d\,(\log \varepsilon^{-1})^2$ random bits.

Problem is tractable for random bit MC methods.