On the power of function values for the approximation problem in various settings

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Problem

Assume that you want to approximate functions from a class F, with error in the L_p-sense.

Are algorithms that are based on arbitrary linear functionals (like Fourier coefficients) better than algorithms that are based on function values?

Observation: For many classes F it turns out that the answer is: *no, not much better*.

Can we prove general results?

Worst Case Setting

Let F be a Banach space of functions such that the $f \mapsto f(x)$ are continuous. Assume $F \subset L_p$, continuous embedding.

Approximate $f \in F$ using linear functionals $L \in F^*$ or function values,

$$A_n(f) = \phi_n(L_1(f), L_2(f), \dots, L_n(f)),$$

where $\phi_n : \mathbb{R}^n \to L_p$ and $L_j \in \Lambda$, where $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$.

Define

$$e_n^{\text{all}-\text{wor}}(F, L_p) = \inf_{A_n \text{ with } L_j \in \Lambda^{\text{all}}} \sup_{\|f\|_F \le 1} \big\| f - A_n(f) \big\|_p$$

and

$$e_n^{\text{std}-\text{wor}}(F, L_p) = \inf_{A_n \text{ with } L_j \in \Lambda^{\text{std}}} \sup_{\|f\|_F \le 1} \big\| f - A_n(f) \big\|_p.$$

Example: Sobolev Spaces, $p = 2$

a) Standard Sobolev spaces $W_2^s([0,1]^d)$ with $2s > d$, known

$$e_n^{\mathrm{all-wor}}(W_2^s([0,1]^d), L_2) \asymp e_n^{\mathrm{std-wor}}(W_2^s([0,1]^d), L_2) \asymp n^{-s/d}.$$

b) Sobolev spaces $W_2^{r,\mathrm{mix}}([0,1]^d)$ with $r > 1/2$, known

$$e_n^{\mathrm{all-wor}}(W_2^{r,\mathrm{mix}}([0,1]^d), L_2) \asymp n^{-r} (\log n)^{(d-1)r},$$

and

$$e_n^{\mathrm{std-wor}}(W_2^{r,\mathrm{mix}}([0,1]^d), L_2) = \mathcal{O}\left(n^{-r} (\log n)^{(d-1)(r+1/2)} \right).$$

Not known whether this extra power $(d-1)/2$ for log is needed.

Rate of Convergence + Power Function

Assume that (c_n) converges to zero. Define its rate of convergence by

$$r(c_n) = \sup\{\, \beta \geq 0 \mid \lim_{n\to\infty} c_n n^\beta = 0\}.$$

For $\alpha > 0$ the rate of convergence of $n^{-\alpha}$ is α.

Compare the rates of $e_n^{\mathrm{all-wor}}(F, L_p)$ and of $e_n^{\mathrm{std-wor}}(F, L_p)$.

Define the power function by

$$\ell^{\,\mathrm{wor-x}}(r,p) := \inf_{F:\, r^{\mathrm{all-wor}}(F,L_p)=r} \frac{r^{\mathrm{std-wor}}(F,L_p)}{r},$$

where $\mathrm{x} \in \{H, B\}$ indicates that all Hilbert spaces $(\mathrm{x} = H)$ or all Banach spaces $(\mathrm{x} = B)$ are taken, for which the rate is r when we use linear functionals.

Double Hilbert Case

Wasilkowski, Woźniakowski (2001):
$r^{\text{all-wor}}(H, L_2) = r > \frac{1}{2}$ implies $r^{\text{std-wor}}(H, L_2) \geq r^{\text{all-wor}}(H, L_2) - \frac{1}{2}$.
Improved by Kuo, Wasilkowski, Woźniakowski (2009) to
$r^{\text{std-wor}}(H, L_2) \geq r - \frac{r}{2r+1} = \frac{2r^2}{2r+1}$.
Case $r \leq 1/2$ was studied in Hinrichs, N., Vybiral (2008): there is a Hilbert space H such that
$r^{\text{all-wor}}(H, L_2) = r$ and $r^{\text{std-wor}}(H, L_2) = 0$.
Hence

$$\ell^{\text{wor-H}}(r, 2) = 0 \quad \text{for all} \quad r \in (0, \tfrac{1}{2}],$$

$$\ell^{\text{wor-H}}(r, 2) \in \left[\frac{2r}{2r+1}, 1\right] \quad \text{for all} \quad r \in (\tfrac{1}{2}, \infty).$$

Open Problem: Suppose that $r > 1/2$. Is it true that $\ell^{\text{wor-H}}(r, 2) = 1$?

Simple Hilbert Case

Result of Tandetzky: Hilbert spaces and arbitrary $p \in [1, \infty)$. For any $r \in (0, \min(\frac{1}{p}, \frac{1}{2})]$ there exists a Hilbert space H continuously embedded in $L_p = L_p([0,1])$ such that

$$r^{\mathrm{all-wor}}(H, L_p) = r \qquad \text{and} \qquad r^{\mathrm{std-wor}}(H, L_p) = 0.$$

Hence the power function is zero over $(0, \min(\frac{1}{p}, \frac{1}{2})]$. We do not know the the power function over $(\min(\frac{1}{p}, \frac{1}{2}), \infty)$. Hence, for $p \neq 2$,

$$\begin{aligned} \ell^{\,\mathrm{wor-H}}(r,p) &= 0 && \text{for all} \quad r \in (0, \min(\tfrac{1}{p}, \tfrac{1}{2})], \\ \ell^{\,\mathrm{wor-H}}(r,p) &\in [0,1] && \text{for all} \quad r \in (\min(\tfrac{1}{p}, \tfrac{1}{2}), \infty). \end{aligned}$$

Only for $p = \infty$ we know more, see N. 88.

$$\ell^{\,\mathrm{wor-H/B}}(r, \infty) \in \left[\frac{r-1}{r}, 1\right] \qquad \text{for all} \quad r > 1.$$

Banach Spaces

We summarize the properties of the power function that can be proved using results from the literature on Sobolev embeddings:

$$\ell^{\mathrm{wor-B}}(r,p) = 0 \qquad r \in (0,1] \text{ and } p \in [1,2],$$

$$\ell^{\mathrm{wor-B}}(r,p) = 0 \qquad r \in (0, \tfrac{1}{2} + \tfrac{1}{p}] \text{ and } p \in (2,\infty),$$

$$\ell^{\mathrm{wor-B}}(r,p) \le 1 - \frac{1}{r}\left(1 - \frac{1}{p}\right) \qquad r > 1 \text{ and } p \in [1,2],$$

$$\ell^{\mathrm{wor-B}}(r,p) \le 1 - \frac{1}{2r} \qquad r > 1 \text{ and } p \in [2,\infty),$$

$$1 - \frac{1}{r} \le \ell^{\mathrm{wor-B}}(r,\infty) \le 1 - \frac{1}{2r} \qquad r > 1.$$

Randomized Setting, double Hilbert case

Consider randomized algorithms and define the error by

$$e^{\mathrm{ran}}(A_n) = \sup_{\|f\|_F \le 1} \left(\mathbb{E}_\omega \|I(f) - A_n(f,\omega)\|_p^2\right)^{1/2}.$$

Compare the rates of convergence

$r^{\mathrm{all-ran}}(F, L_p) = r\left(e_n^{\mathrm{all-ran}}(F, L_p)\right)$ and $r^{\mathrm{std-ran}}(F, L_p) = r\left(e_n^{\mathrm{std-ran}}(F, L_p)\right)$.

Result of Wasilkowski, Woźniakowski (2007):
For arbitrary Hilbert spaces $I : H \to L_2(\Omega)$

$$r^{\mathrm{all-ran}}(H, L_2) = r^{\mathrm{std-ran}}(H, L_2).$$

Therefore

$$\ell^{\,\mathrm{ran-H}}(r,2) = 1 \qquad \text{for all} \quad r > 0.$$

It was known before that also

$$r^{\mathrm{all-ran}}(H, L_2) = r^{\mathrm{all-wor}}(H, L_2).$$

Randomized Setting, other cases

Assume $p > 2$, consider $I : W_2^r([0,1]) \to L_p([0,1])$.
Mathé (1991): With Λ^{all} obtain optimal order n^{-r}.
Heinrich (2008): With Λ^{std} the optimal order is $n^{-r+1/2-1/p}$.

Hence

$$\ell^{\,\mathrm{ran-H}}(r,p) \le \frac{r - 1/2 + 1/p}{r} \qquad \text{if} \quad r \ge 1 \quad \text{and} \quad p > 2.$$

Open Problem:
Study the power function in the randomized setting for the Hilbert case with $p \in [1,2)$ and for the Banach case for all $p \in [1,\infty]$.

Average Case Setting with a Gaussian Measure

Let F be a separable Banach space equipped with a zero mean Gaussian measure μ. Average error

$$e^{\rm avg}(A) = \left(\int_F \|f - A(f)\|_p^2 \, {\rm d}\mu(f) \right)^{1/p}.$$

Define the minimal nth average case errors $e_n^{\rm all-avg}(F, L_p)$, $e_n^{\rm std-avg}(F, L_p)$ and the power function $\ell^{\rm avg-H/B}$.

Results known for $p = 2$.

Wasilkowski, Woźniakowski (2008): Let $I : F \to L_2(\Omega)$ be arbitrary. Then

$$r^{\rm all-avg}(F, L_2) = r^{\rm std-avg}(F, L_2).$$

Therefore

$$\ell^{\rm avg-B}(r, 2) = 1 \qquad \text{for all} \quad r > 0.$$