Randomly permuted and random-started Halton sequences

Giray Ökten
Department of Mathematics, Florida State University
okten@math.fsu.edu

August 11, 2010

1 Random start (permuted) Halton sequences

Van der Corput and Halton sequences The nth term of the van der Corput sequence, $\phi_b(n)$, in base b, is defined as follows: First, write the base b expansion of n:

$$n = (a_k \cdots \ a_1 a_0)_b = a_0 + a_1 b + ... + a_k b^k,$$

then compute

$$\phi_b(n) = (.a_0 a_1 \cdots a_k)_b = \frac{a_0}{b} + \frac{a_1}{b^2} + ... + \frac{a_k}{b^{k+1}}. \tag{1}$$

The Halton sequence in the bases $b_1, ..., b_s$ is $(\phi_{b_1}(n), ..., \phi_{b_s}(n))_{n=1}^{\infty}$. This is a uniformly distributed mod 1 (u.d. mod 1) sequence if the bases are relatively prime. In practice, b_i is usually chosen as the ith prime number.

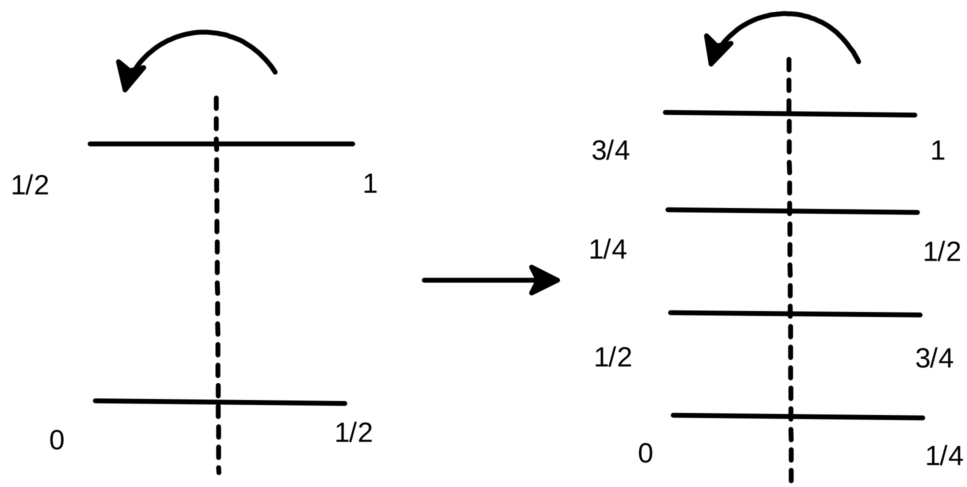
Permuted sequences The permuted van der Corput sequence generalizes (1) as

$$\phi_b(n) = \frac{\sigma(a_0)}{b} + \frac{\sigma(a_1)}{b^2} + ... + \frac{\sigma(a_k)}{b^{k+1}}$$

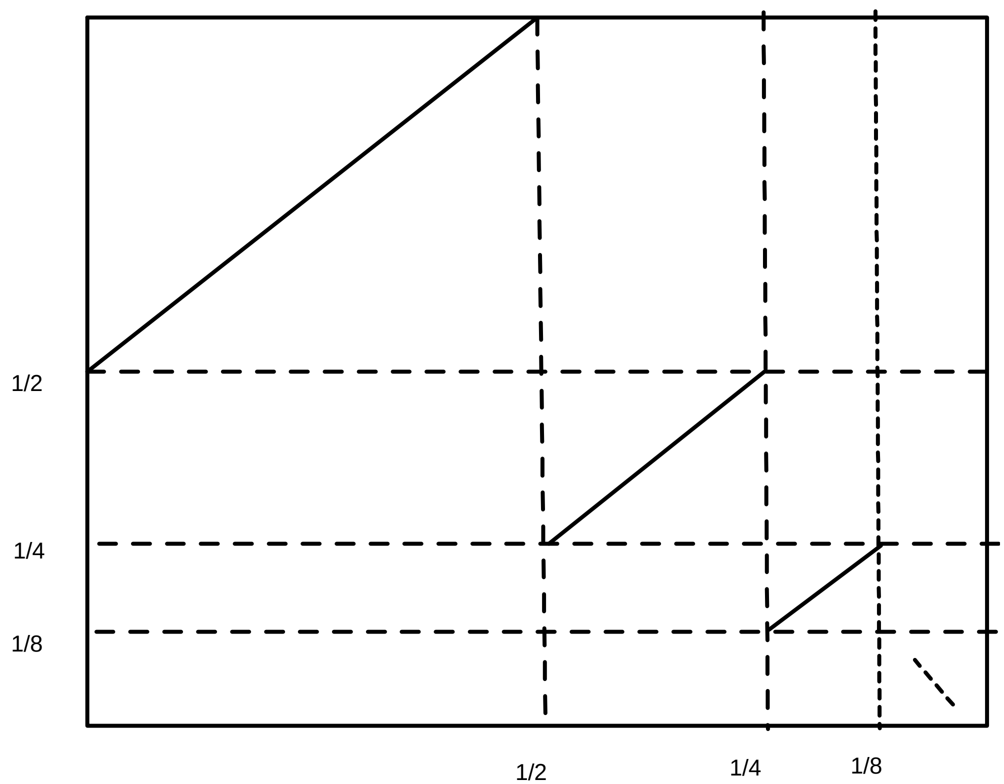
where σ is a permutation on the digit set $\{0, ..., b-1\}$. The scrambled Halton sequence is obtained from scrambled van der Corput sequences in the usual way. A further generalization of the above equation, first considered by Faure, is

$$\phi_b(n) = \frac{\sigma_1(a_0)}{b} + \frac{\sigma_2(a_1)}{b^2} + ... + \frac{\sigma_k(a_{k-1})}{b^k}$$

Von Neumann-Kakutani transformation An ergodic and measure-preserving transformation $T : [0,1) \to [0,1)$, constructed inductively, by a *splitting and stacking* process.



Plotting $T(x) = \lim_{n \to \infty} T_n(x)$; see T_1, T_2 and T_3 below:



Notes 1. If $\mathbf{b} = (b_1, ..., b_s)$ is a vector of positive integers that are pairwise relatively prime and $\mathbf{x} = (x_1, ..., x_s)$ is a vector in $[0, 1)^s$, then the s-dimensional von Neuman-Kakutani transformation is given by

$$\mathbf{T_b}(\mathbf{x}) = \left(T_{b_1}(x_1), ..., T_{b_s}(x_s)\right). \tag{2}$$

Lambert (1982) observed that the orbit of $(0, ..., 0)$ under $\mathbf{T_b}$ gives the Halton sequence in bases $b_1, ..., b_s$.

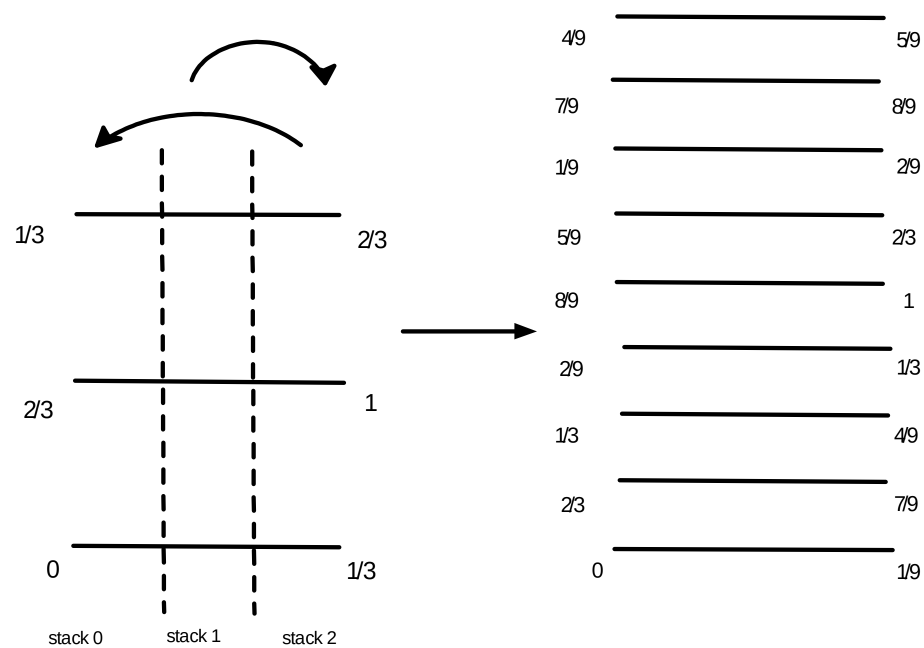
2. The orbit of any vector $\mathbf{x}$ in $[0, 1)^s$ under $\mathbf{T_b}$, $\{\mathbf{T}_\mathbf{b}^n(\mathbf{x})\}_{n=1}^{\infty}$, can be used for numerical integration. Struckmeier (1995) proposed independently selecting random starting values $\mathbf{x}$ in (2).

3. Wang & Hickernell (2000) gave a theoretical justification for randomly selecting starting values, and introduced an RQMC method: random-start Halton sequences.

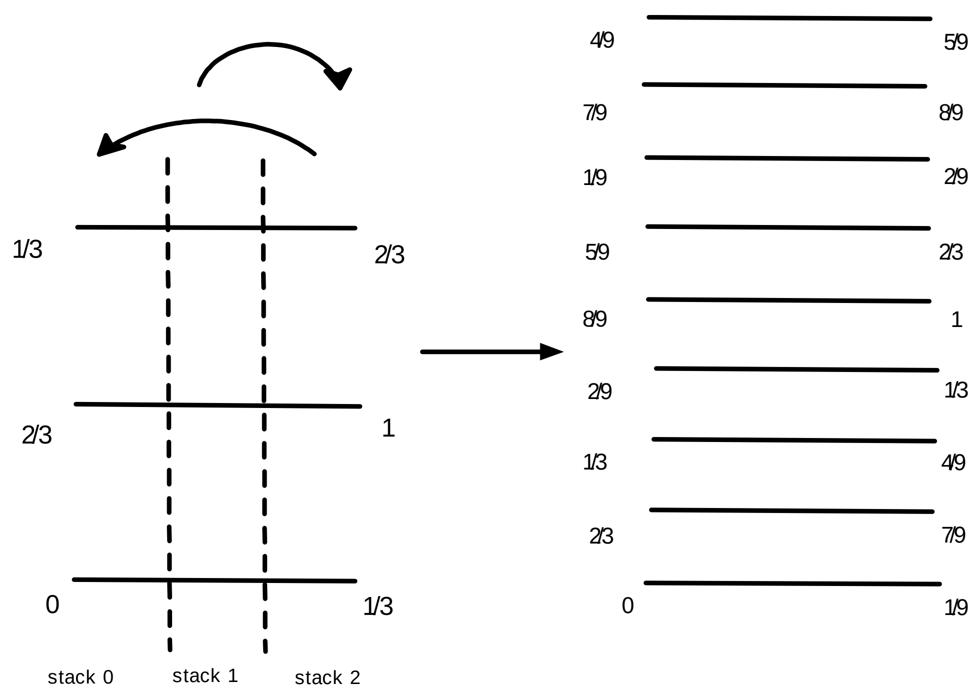
Question Can we represent permuted Halton sequences using the von Neumann-Kakutani transformation? (Why do we want this?)

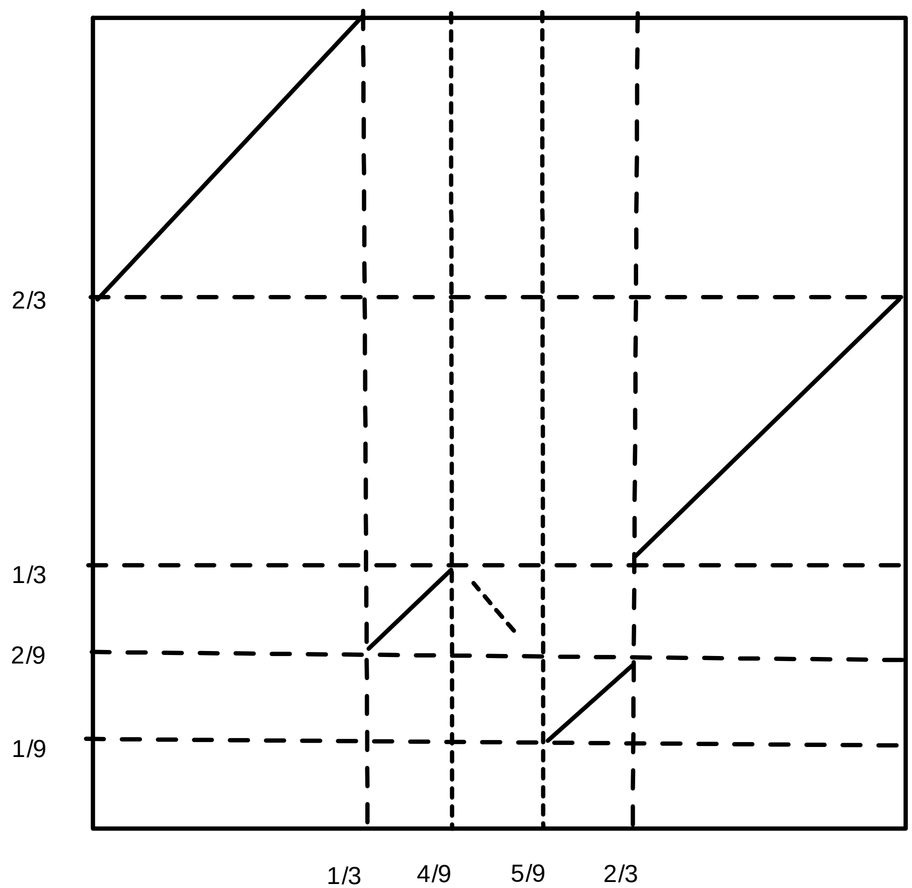
A generalized splitting and stacking process Let $b = 3$ and all permutations equal to $\sigma = (0\ 2\ 1)$ (i.e., $\sigma(0) = 0, \sigma(1) = 2, \sigma(2) = 1$)

The first ladder is obtained by stacking $\underbrace{[0, 1/3)}_{0}, \underbrace{[1/3, 2/3)}_{1}, \underbrace{[2/3, 1)}_{2}$ in the order $\sigma(0) = 0 \prec \sigma(1) = 2 \prec \sigma(2) = 1$



The second ladder is obtained by splitting the first ladder into thirds, labeling the resulting stacks as $0, 1, 2$ (from left to right), and then stacking in the order $0 \prec 2 \prec 1$.




2/3
1/3
2/9
1/9
1/3
4/9
5/9
2/3

Remark The orbit of 0 under the mapping T in the above example is {0, 2/3, 1/3, 2/9, 8/9, 5/9, 1/9, 7/9, 4/9,...}, which is the permuted van der Corput sequence in base 3 with all permutations equal to $\sigma = (0\ 2\ 1)$ (Braaten and Weller permutation)

Theorem 1. The resulting transformation induced by the generalized (normal) splitting and stacking process $T : [0, 1) \to [0, 1)$ is ergodic and measure preserving

2. The orbit of 0 under a generalized splitting and stacking process in base b with permutations $\sigma_1, \sigma_2, ...$, coincides with the permuted van der Corput sequence in base b with the same permutations

Random-start permuted Halton sequences Define a random variable θ_N as

$$\theta_N(\mathbf{X}) = \frac{1}{N}\sum_{i=0}^{N-1} f(\mathbf{T}^i(X))$$

where f is a function defined on $[0,1)^s$, $I = \int_{[0,1)^s} f(x)dx$, $\mathbf{X}$ is $U(0,1)^s$, and $\mathbf{T} = (T_{b_1}, ..., T_{b_s})$ is a generalized von Neumann-Kakutani transformation

Theorem $\theta_N(\mathbf{X})$ is an unbiased estimator for I

Theorem

- Lapeyre & Pagès (1989): Consider the sequence $\{\mathbf{T}^n(\mathbf{x})\}_{n=0}^{\infty}$ where all the permutations used in the splitting and stacking process are the identity. Then

$$D_N^*(\mathbf{T}^n(\mathbf{x})) \le \frac{1}{N}\left[1 + \prod_{i=1}^{s}(b_i - 1)\frac{\log b_i N}{\log b_i}\right]$$

- Atanassov (2004): Consider the sequence $\{\mathbf{T}^n(\mathbf{0})\}_{n=0}^{\infty}$ where arbitrary permutations are used in the scrambled splitting and stacking process. Then

$$D_N^*(\mathbf{T}^n(\mathbf{0})) \leq \frac{1}{N}\frac{1}{s!}\prod_{i=1}^{s}\frac{b_i - 1}{\log b_i}(\log N)^s + O(N^{-1}(\log N)^{s-1}).$$

Conjecture $D_N^*(\mathbf{T}^n(\mathbf{x}))$ is $O(N^{-1}(\log N)^s)$ for arbitrary permutations used in the splitting and stacking process, arbitrary $\mathbf{x}$, and any relatively prime bases $b_1, ..., b_s$

Remark If this conjecture is true, then

$$\sigma^2(\theta_N(\mathbf{X})) = O(N^{-2}(\log N)^{2s})$$

2 Randomly permuted Halton sequences

A well known defect of the Halton sequence: *high correlation between higher bases*

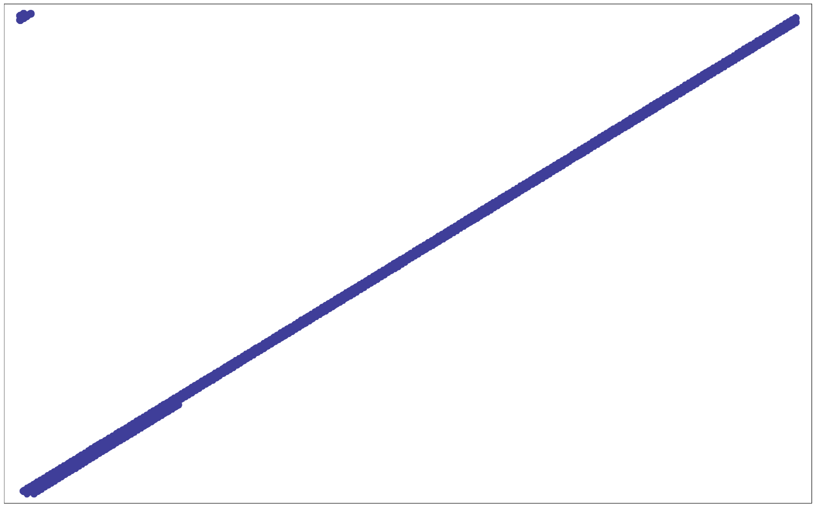


Figure 1: 500 Halton vectors in bases 227 and 229

Remedy: Permutations by Braaten & Weller, Atanassov, Faure, Chi & Mascagni & Warnock, Faure & Lemieux, Kocis & Whitten, Tuffin, Vandewoestyne & Cool, Warnock.

Question Do these sequences avoid the phenomenon of *high correlation between higher bases?*

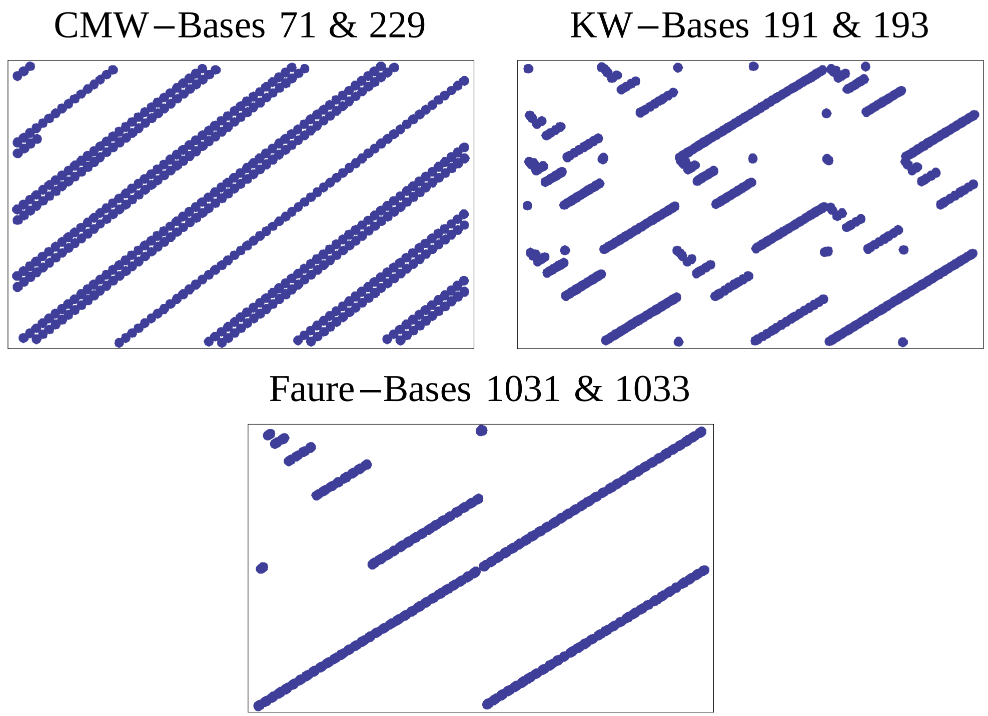
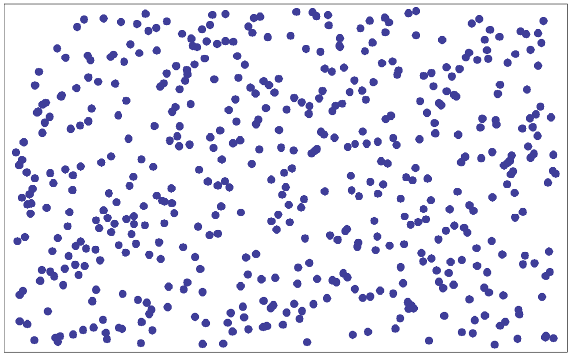


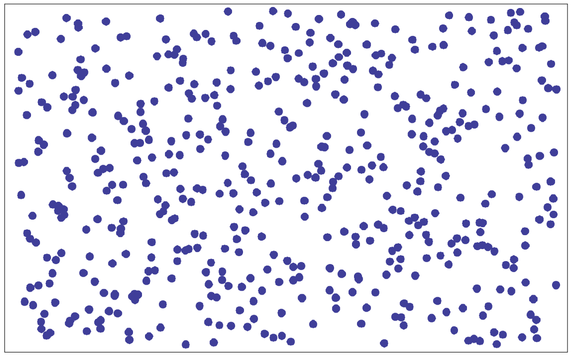
Figure 2: 500 vectors from digit permuted Halton sequences

Question What if we used a randomly picked permutation, from the space of all permutations, to permute the digits of the Halton sequence? How would this approach compare with the existing deterministic permutations?

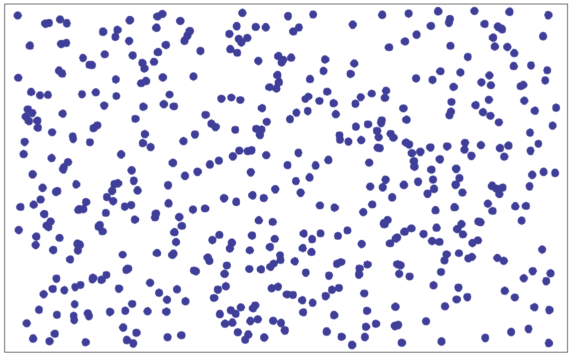
Random permutation −Bases 71 & 229



Random permutation −Bases 191 & 193



Random permutation −Bases 1031 & 1033



Numerical comparison of random & deterministic permuted Halton sequences:

- Compute star-discrepancy: a lower-bound using a genetic algorithm by Shah, and an upper bound using an algorithm by Thiémard

- Compare sequences when applied to test problems

2.1 Discrepancy results

2.1.1 Computing the star discrepancy of permuted Halton sequences

Case A: bases are the first ten primes.
Case B: bases are the ith prime numbers where $i \in \{11, 17, 21, 22, 24, 29, 31, 35, 37, 40\}$
Case C: $i \in \{41, 42, 43, 44, 45, 46, 47, 48, 49, 50\}$
Case D: $i \in \{43, 44, 49, 50, 76, 77, 135, 136, 173, 174\}$

D^*_{100}	Case A	Case B	Case C	Case D
Halton	$(0.251, 0.387)$	$(0.769, 0.962)$	$(0.910, 1.000)$	$(0.860, 1.000)$
Reverse	$(0.244, 0.392)$	$(0.429, 0.569)$	$(0.485, 0.640)$	$(0.903, 0.927)$
Faure	$(0.157, 0.324)$	$(0.238, 0.395)$	$(0.209, 0.388)$	$(0.360, 0.555)$
KW	$(0.171, 0.331)$	$(0.285, 0.451)$	$(0.212, 0.378)$	$(0.419, 0.573)$
CMW	$(0.184, 0.337)$	$(0.198, 0.364)$	$(0.548, 0.683)$	N/A
Random	$(0.182, 0.345)$	$(0.212, 0.373)$	$(0.259, 0.444)$	$(0.294, 0.437)$
C.I.	$(0.177, 0.187)$	$(0.205, 0.220)$	$(0.253, 0.267)$	$(0.288, 0.303)$

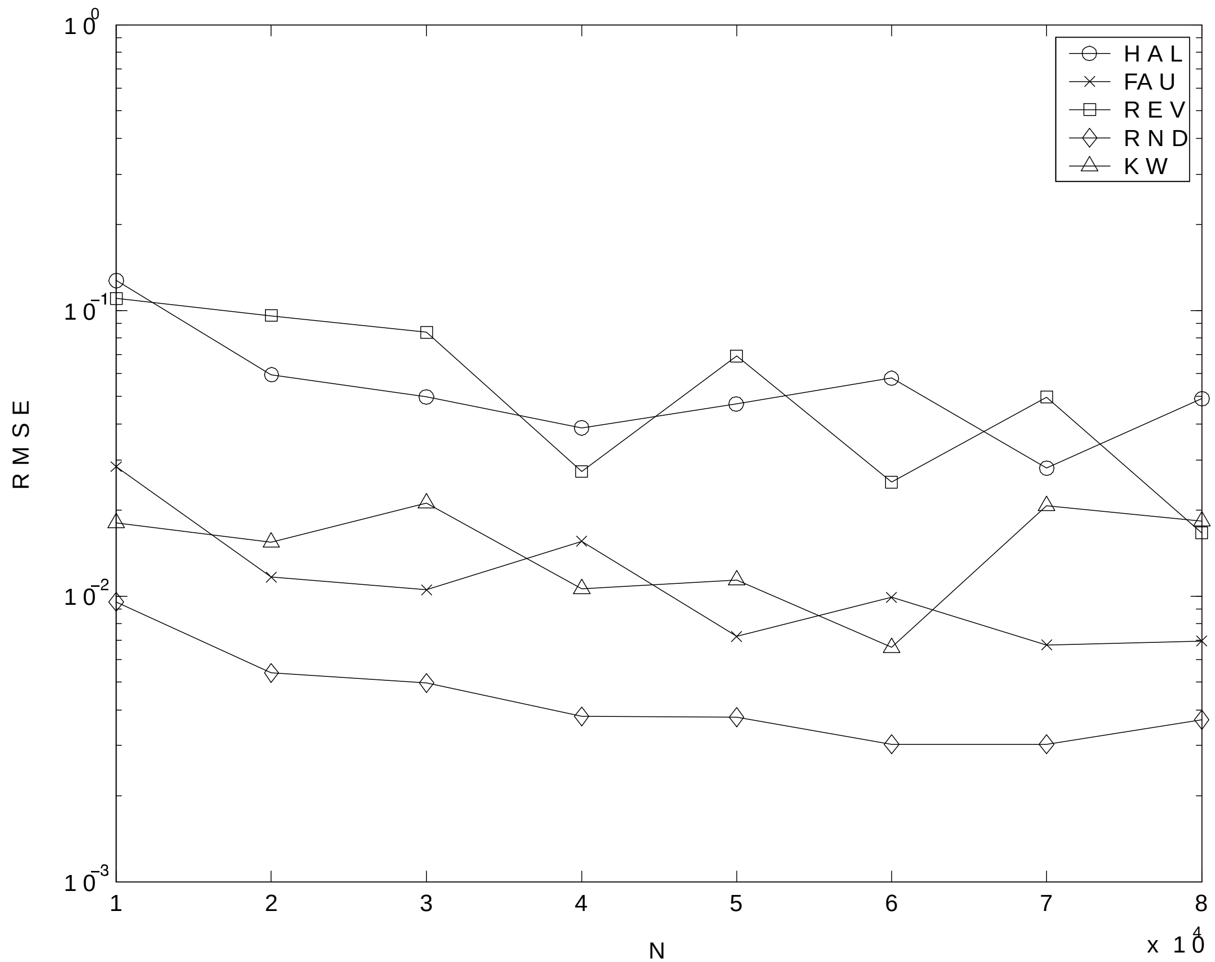
2.1.2 Computing the star discrepancy of random-start permuted Halton sequences

D^*_{100}	Lower	Upper
MC (tt800)	0.1934	0.4059
Faure	0.1846	0.3323
CMW	0.1835	0.3236
Random	0.1843	0.3165

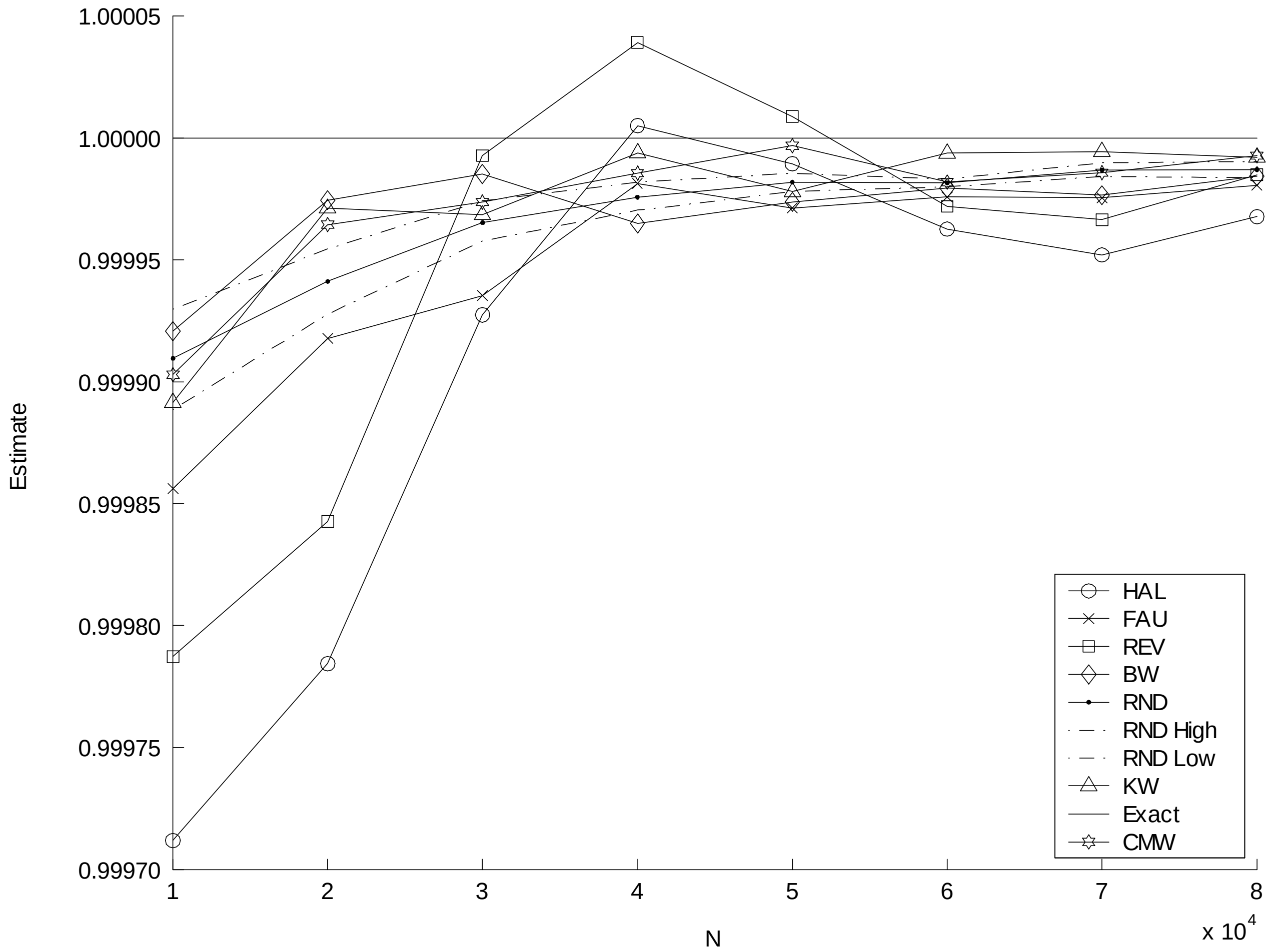
(Lower bounds are the maximum of ten GA runs)

2.2 Numerical integration

$$f(x_1, ..., x_s) = \prod \frac{|4x_i - 2| + a_i}{1 + a_i}$$



Numerical Integration: Case D



Increasing importance case: $a_i = (10 - i)^2$

2.3 Pricing a ratchet option

N	LinScrF	CMW	Faure	Random	MC
10K	21	12	31	19	154
20K	11	8.2	13	10	106
30K	7.9	4.8	7.6	8.1	85
40K	6.6	5.2	6.7	6.2	73
50K	5.5	4.7	3.9	4.4	64
60K	6.4	4.4	6.0	4.6	63
70K	4.6	3.8	5.1	4.8	49
80K	5.2	3.9	3.1	4.2	51

RMSE ($\times 10^{-3}$)