Randomly permuted and random-started Halton sequences

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1 Random start (permuted) Halton sequences

Van der Corput and Halton sequences The \( n \)th term of the van der Corput sequence, \( \phi_b(n) \), in base \( b \), is defined as follows: First, write the base \( b \) expansion of \( n \):

\[
n = \left( a_k \cdots a_1 a_0 \right)_b = a_0 + a_1 b + \cdots + a_k b^k,
\]

then compute

\[
\phi_b(n) = (.a_0 a_1 \cdots a_k)_b = \frac{a_0}{b} + \frac{a_1}{b^2} + \cdots + \frac{a_k}{b^{k+1}}.
\] (1)

The Halton sequence in the bases \( b_1, \ldots, b_s \) is \((\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n=1}^\infty\). This is a uniformly distributed mod 1 (u.d. mod 1) sequence if the bases are relatively prime. In practice, \( b_i \) is usually chosen as the \( i \)th prime number.
**Permuted sequences** The permuted van der Corput sequence generalizes (1) as

\[
\phi_b(n) = \frac{\sigma(a_0)}{b} + \frac{\sigma(a_1)}{b^2} + \ldots + \frac{\sigma(a_k)}{b^{k+1}}
\]

where \(\sigma\) is a permutation on the digit set \(\{0, \ldots, b-1\}\). The scrambled Halton sequence is obtained from scrambled van der Corput sequences in the usual way. A further generalization of the above equation, first considered by Faure, is

\[
\phi_b(n) = \frac{\sigma_1(a_0)}{b} + \frac{\sigma_2(a_1)}{b^2} + \ldots + \frac{\sigma_k(a_{k-1})}{b^k}
\]
**Von Neumann-Kakutani transformation**  An ergodic and measure-preserving transformation $T : [0, 1) \rightarrow [0, 1)$, constructed inductively, by a *splitting and stacking* process.

![Diagram](image)
Plotting $T(x) = \lim_{n \to \infty} T_n(x)$; see $T_1$, $T_2$ and $T_3$ below:
Notes 1. If \( b = (b_1, \ldots, b_s) \) is a vector of positive integers that are pairwise relatively prime and \( x = (x_1, \ldots, x_s) \) is a vector in \([0, 1)^s\), then the \( s \)-dimensional von Neuman-Kakutani transformation is given by

\[
T_b(x) = \left(T_{b_1}(x_1), \ldots, T_{b_s}(x_s)\right).
\]  

(2)

Lambert (1982) observed that the orbit of \((0, \ldots, 0)\) under \( T_b \) gives the Halton sequence in bases \( b_1, \ldots, b_s \).

2. The orbit of any vector \( x \) in \([0, 1)^s\) under \( T_b \), \( \{T_b^n(x)\}_{n=1}^{\infty} \), can be used for numerical integration. Struckmeier (1995) proposed independently selecting random starting values \( x \) in (2).

Question  Can we represent permuted Halton sequences using the von Neumann-Kakutani transformation? (Why do we want this?)
A generalized splitting and stacking process  Let $b = 3$ and all permutations equal to $\sigma = (0 \ 2 \ 1)$ (i.e., $\sigma(0) = 0, \sigma(1) = 2, \sigma(2) = 1$)

The first ladder is obtained by stacking $[0, 1/3), [1/3, 2/3), [2/3, 1)$ in the order $\sigma(0) = 0 \prec \sigma(1) = 2 \prec \sigma(2) = 1$
The second ladder is obtained by splitting the first ladder into thirds, labeling the resulting stacks as 0, 1, 2 (from left to right), and then stacking in the order \(0 < 2 < 1\).
Remark  The orbit of 0 under the mapping $T$ in the above example is \{0, 2/3, 1/3, 2/9, 8/9, 5/9, 1/9, 7/9, 4/9,...\}, which is the permuted van der Corput sequence in base 3 with all permutations equal to $\sigma = (0 \ 2 \ 1)$ (Braaten and Weller permutation)

Theorem  1. The resulting transformation induced by the generalized (normal) splitting and stacking process $T : [0, 1) \to [0, 1)$ is ergodic and measure preserving

2. The orbit of 0 under a generalized splitting and stacking process in base $b$ with permutations $\sigma_1, \sigma_2, ..., \sigma_n$, coincides with the permuted van der Corput sequence in base $b$ with the same permutations
Random-start permuted Halton sequences  Define a random variable $\theta_N$ as

$$\theta_N(X) = \frac{1}{N} \sum_{i=0}^{N-1} f(T_i(X))$$

where $f$ is a function defined on $[0, 1)^s$, $I = \int_{[0,1)^s} f(x) dx$, $X$ is $U(0, 1)^s$, and $T = (T_{b_1}, ..., T_{b_s})$ is a generalized von Neumann-Kakutani transformation.

**Theorem** $\theta_N(X)$ is an unbiased estimator for $I$

**Theorem** Lapeyre & Pagès (1989): Consider the sequence $\{T^n(x)\}_{n=0}^{\infty}$ where all the permutations used in the splitting and stacking process are the identity. Then

$$D_N^*(T^n(x)) \leq \frac{1}{N} \left[ 1 + \prod_{i=1}^{s} (b_i - 1) \frac{\log b_i N}{\log b_i} \right]$$
• Atanassov (2004): Consider the sequence \{T^n(0)\}_{n=0}^{\infty} where arbitrary permutations are used in the scrambled splitting and stacking process. Then

\[ D_N^*(T^n(0)) \leq \frac{1}{N^s!} \prod_{i=1}^{s} \frac{b_i - 1}{\log b_i} (\log N)^s + O(N^{-1}(\log N)^{s-1}). \]

**Conjecture** \( D_N^*(T^n(x)) \) is \( O(N^{-1}(\log N)^s) \) for arbitrary permutations used in the splitting and stacking process, arbitrary \( x \), and any relatively prime bases \( b_1, \ldots, b_s \)

**Remark** If this conjecture is true, then

\[ \sigma^2(\theta_N(X)) = O(N^{-2}(\log N)^{2s}) \]
2 Randomly permuted Halton sequences

A well known defect of the Halton sequence: *high correlation between higher bases*

Figure 1: 500 Halton vectors in bases 227 and 229
**Remedy:** Permutations by Braaten & Weller, Atanassov, Faure, Chi & Mascagni & Warnock, Faure & Lemieux, Kocis & Whitten, Tuffin, Vandewoestyne & Cool, Warnock.

**Question**  Do these sequences avoid the phenomenon of *high correlation between higher bases*?
Figure 2: 500 vectors from digit permuted Halton sequences
**Question** What if we used a randomly picked permutation, from the space of all permutations, to permute the digits of the Halton sequence? How would this approach compare with the existing deterministic permutations?
Random permutation — Bases 71 & 229

Random permutation — Bases 191 & 193

Random permutation — Bases 1031 & 1033
Numerical comparison of random & deterministic permuted Halton sequences:

- Compute star-discrepancy: a lower-bound using a genetic algorithm by Shah, and an upper bound using an algorithm by Thiémard

- Compare sequences when applied to test problems
2.1 Discrepancy results

2.1.1 Computing the star discrepancy of permuted Halton sequences

Case A: bases are the first ten primes.
Case B: bases are the $i$th prime numbers where $i \in \{11, 17, 21, 22, 24, 29, 31, 35, 37, 40\}$
Case C: $i \in \{41, 42, 43, 44, 45, 46, 47, 48, 49, 50\}$
Case D: $i \in \{43, 44, 49, 50, 76, 77, 135, 136, 173, 174\}$
<table>
<thead>
<tr>
<th>$D_{100}^*$</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
<th>Case D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Halton</td>
<td>(0.251, 0.387)</td>
<td>(0.769, 0.962)</td>
<td>(0.910, 1.000)</td>
<td>(0.860, 1.000)</td>
</tr>
<tr>
<td>Reverse</td>
<td>(0.244, 0.392)</td>
<td>(0.429, 0.569)</td>
<td>(0.485, 0.640)</td>
<td>(0.903, 0.927)</td>
</tr>
<tr>
<td>Faure</td>
<td>(0.157, 0.324)</td>
<td>(0.238, 0.395)</td>
<td>(0.209, 0.388)</td>
<td>(0.360, 0.555)</td>
</tr>
<tr>
<td>KW</td>
<td>(0.171, 0.331)</td>
<td>(0.285, 0.451)</td>
<td>(0.212, 0.378)</td>
<td>(0.419, 0.573)</td>
</tr>
<tr>
<td>CMW</td>
<td>(0.184, 0.337)</td>
<td>(0.198, 0.364)</td>
<td>(0.548, 0.683)</td>
<td>N/A</td>
</tr>
<tr>
<td>Random</td>
<td>(0.182, 0.345)</td>
<td>(0.212, 0.373)</td>
<td>(0.259, 0.444)</td>
<td>(0.294, 0.437)</td>
</tr>
<tr>
<td>C.I.</td>
<td>(0.177, 0.187)</td>
<td>(0.205, 0.220)</td>
<td>(0.253, 0.267)</td>
<td>(0.288, 0.303)</td>
</tr>
</tbody>
</table>
2.1.2 Computing the star discrepancy of random-start permuted Halton sequences

<table>
<thead>
<tr>
<th>$D_{100}^*$</th>
<th>Lower</th>
<th>Upper</th>
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</thead>
<tbody>
<tr>
<td>MC (tt800)</td>
<td>0.1934</td>
<td>0.4059</td>
</tr>
<tr>
<td>Faure</td>
<td>0.1846</td>
<td>0.3323</td>
</tr>
<tr>
<td>CMW</td>
<td>0.1835</td>
<td>0.3236</td>
</tr>
<tr>
<td>Random</td>
<td>0.1843</td>
<td>0.3165</td>
</tr>
</tbody>
</table>

(Lower bounds are the maximum of ten GA runs)
2.2 Numerical integration

\[ f(x_1, \ldots, x_s) = \prod \frac{|4x_i - 2| + a_i}{1 + a_i} \]
Numerical Integration: Case D
Increasing importance case: $a_i = (10 - i)^2$
## 2.3 Pricing a ratchet option

<table>
<thead>
<tr>
<th>( N )</th>
<th>LinScrF</th>
<th>CMW</th>
<th>Faure</th>
<th>Random</th>
<th>MC</th>
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<tbody>
<tr>
<td>10K</td>
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<td>31</td>
<td>19</td>
<td>154</td>
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<tr>
<td>20K</td>
<td>11</td>
<td>8.2</td>
<td>13</td>
<td>10</td>
<td>106</td>
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<tr>
<td>30K</td>
<td>7.9</td>
<td>4.8</td>
<td>7.6</td>
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</tr>
<tr>
<td>40K</td>
<td>6.6</td>
<td>5.2</td>
<td>6.7</td>
<td>6.2</td>
<td>73</td>
</tr>
<tr>
<td>50K</td>
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<td>3.9</td>
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</tr>
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<td>70K</td>
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</tr>
<tr>
<td>80K</td>
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<td>3.1</td>
<td>4.2</td>
<td>51</td>
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</tbody>
</table>

RMSE \( (\times 10^{-3}) \)