# Approximating a Geometric fractional Brownian motion and related processes via discrete Wick calculus

a joint work with Christian Bender - Saarland University

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15.8.2010

# Introduction

The stochastic exponential $\exp\left(B_t - \frac{1}{2}t\right)$ solves the Doléans-Dade SDE

$$dS_t = S_t dB_t \ , \ S_0 = 1$$

in terms of the Itô integral.

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in terms of the Itô integral.
For a fractional Brownian motion $B_t^H$ the exponential $\exp\left(B_t^H - \frac{1}{2}t^{2H}\right)$ generalizes the stochastic exponential and solves the fractional Doléans-Dade SDE

$$dS_t = S_t d^{\diamond} B_t^H \ , \ S_0 = 1$$

in terms of the fractional Wick-Itô integral.

There is the representation

$$\exp\left(B_t^H - \frac{1}{2}t^{2H}\right) =: \exp^{\diamond}\left(B_t^H\right) = \sum_{n=0}^{\infty} \frac{1}{n!}(B_t^H)^{\diamond n}$$

where $\diamond$ denotes the **Wick product**.

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where $\diamond$ denotes the **Wick product**.
More generally, we consider a linear system of SDEs,

$$\begin{aligned} dX_t &= (A_1 X_t + A_2 Y_t)\, d^{\diamond} B_t^H, & X_0 &= x_0, \\ dY_t &= (B_1 X_t + B_2 Y_t)\, d^{\diamond} B_t^H, & Y_0 &= y_0. \end{aligned} \tag{1}$$

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One can obtain Wick power series expansions for the solution of this system, too.

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**Aim**

Approximate the solution of 1.

1. Preliminaries
2. The approximation theorems
   - The approximations theorems
   - Examples
3. Convergence
   - Walsh decompositions and $L^2$-estimates
   - Hermite recursion
   - Tightness
4. Generalizations

# Fractional Brownian motion

## Definition

A **fractional Brownian motion (fBM)** $B^H$ with Hurst parameter $H \in (0,1)$ is a continuous zero mean Gaussian process in $\mathbb{R}$ with stationary increments and covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

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- $B_t^H$ is not a semimartingale for $H \neq \frac{1}{2}$.
- Long range dependence for $H \in (\frac{1}{2}, 1)$.
- We consider only $H \in (\frac{1}{2}, 1)$.

Representation on the interval $[0,1]$ based on works by Molchan and Golosov (cf. Nualart):

$$B_t^H = \int_0^t z(t,s)dB_s$$

with the deterministic kernel

$$z(t,s) = c_H(H - \frac{1}{2})s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du$$

with the constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$$

where $\Gamma$ is the Gamma function and $z(t,s) = 0$ whenever $t \leq s$.

# Sottinens approximation

- Donsker's theorem: $B_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n$ converges weakly to a Brownian motion, where $\xi_i^n$ are i.i.d. with $P(\xi_i^n = 1) = P(\xi_i^n = -1) = \frac{1}{2}$.

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**Theorem (Sottinen 2001)**

*If* $z^{(n)}(t,s) := n \int_{s-\frac{1}{n}}^{s} z(\frac{\lfloor nt \rfloor}{n}, u)du$, *then*

$$B_t^{H,n} := \int_0^t z^{(n)}(t,s)dB_s^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt \rfloor}{n}, s)ds \frac{1}{\sqrt{n}} \xi_i^n \quad (2)$$

*converges weakly to a fractional Brownian motion* $(B_t^H)_{t\in[0,1]}$.

# Wick produckt

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For a zero mean Gaussian random variable $\Phi$ the **Wick exponential** is defined as

$$\exp^{\diamond}(\Phi) := \exp\left(\Phi - \frac{1}{2}\mathbf{E}[|\Phi|^2]\right)$$

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- $\diamond$ can be extended to a larger class of random elements by density arguments.
- $\diamond$ is not a pointwise operation!

# Discrete Wick calculus

$$B_t^{H,n} = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\frac{\lfloor nt \rfloor}{n}, s) ds \frac{1}{\sqrt{n}} \xi_i^n$$

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## Definition

For any fixed $n \in \mathbb{N}$ the **discrete Wick product** is defined as

$$\prod_{i \subset A} \xi_i^n \diamond_n \prod_{i \subset B} \xi_i^n \;:=\; \begin{cases} \prod_{i \subset A \cup B} \xi_i^n & \text{if } A \cap B = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $A, B \subseteq \{1, \ldots, n\}$.

# Walsh decomposition

We denote by $\mathcal{F}_n := \sigma\left(\xi_1^n, \xi_2^n, \ldots, \xi_n^n\right)$ the $\sigma$-field generated by the Bernoulli variables.

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Every $X \in L^2(\Omega, \mathcal{F}_n, P)$ has a unique expansion, called the **Walsh decomposition**,

$$X = \sum_{A \subseteq \{1,\ldots,n\}} x_A^n \Psi_A^n,$$

where $\Psi_A^n := \prod_{i \in A} \xi_i^n$, $x_A^n \in \mathbb{R}$.

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- For $X, Y \in L^2(\Omega, \mathcal{F}_n, P)$ , $\mathbf{E}\left[XY\right] = \sum_{A \subseteq \{1,\ldots,n\}} x_A^n y_A^n$.

# Hermite polynomials

## Definition

The **Hermite polynomial of degree** $n \in \mathbb{N}$ **with parameter** $p$ is defined as

$$h_p^n(x) := (-p)^n \exp\left(\frac{x^2}{2p}\right) \frac{d^n}{dx^n} \exp\left(\frac{-x^2}{2p}\right).$$

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Hermite recursion formula

$$h_p^{n+1}(x) = x\, h_p^n(x) - np\, h_p^{n-1}(x). \tag{4}$$

- By the fractional Itô formula we have

$$d(B_t^H)^{\diamond k} \;=\; k(B_t^H)^{\diamond k-1} d^{\diamond} B_t^H, \qquad (B_0^H)^{\diamond k} = \mathbf{1}_{\{k=0\}}. \tag{5}$$

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- For any Gaussian random variable $\Phi \sim \mathcal{N}(0, \sigma)$ and all $n \in \mathbb{N}$,

$$\Phi^{\diamond n} = h^n_{\sigma^2}(\Phi). \quad (6)$$

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- $\exp\left(B_t^H - \frac{1}{2}t^{2H}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}(B_t^H)^{\diamond n}$
- $\exp^{\diamond}\left(B_t^H\right)$ solves the fractional Doléans-Dade SDE.

The coefficients of the solution of

$$dX_t = (A_1 X_t + A_2 Y_t)\, d^{\diamond} B_t^H, \qquad X_0 = x_0,$$
$$dY_t = (B_1 X_t + B_2 Y_t)\, d^{\diamond} B_t^H, \qquad Y_0 = y_0,$$

$$X_t = \sum_{k=0} \frac{a_k}{k!} \left(B_t^H\right)^{\diamond k}, \qquad Y_t = \sum_{k=0} \frac{b_k}{k!} \left(B_t^H\right)^{\diamond k}, \tag{7}$$

can be obtained recursively via (5) to be

$$a_0 = x_0,\ b_0 = y_0,\ a_k = A_1 a_{k-1} + A_2 b_{k-1},\ b_k = B_1 a_{k-1} + B_2 b_{k-1}.$$

# The approximation theorems

## Theorem

*Suppose*

1. $\lim_{n\to\infty} a_{n,k} = a_k$ *exists for all* $k \in \mathbb{N}$.
2. *There exists a* $C \in \mathbb{R}_+$, *so that* $|a_{n,k}| \leq C^k$ *for all* $n, k \in \mathbb{N}$.

*Then the sequence of processes* $\sum_{k=0}^{n} \frac{a_{n,k}}{k!} \left(B^{H,n}\right)^{\diamond_n k}$ *converges weakly to the Wick power series* $\sum_{k=0}^{\infty} \frac{a_k}{k!} \left(B^{H}\right)^{\diamond k}$ *in the Skorokhod space* $D([0,1],\mathbb{R})$.

The recursive system of Wick difference equations,

$$U_l^{k,n} = U_{l-1}^{k,n} + kU_{l-1}^{k-1,n} \diamond_n \left( B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), U_l^{0,n} = 1, \ U_0^{k,n} = 0 \quad (8)$$

for all $l = 1, \ldots, n$ and $k \in \mathbb{N}$.

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- $U^{0,n} = 1 = \left(B^{H,n}\right)^{\diamond_n 0}$ and $U^{1,n} = \left(B^{H,n}\right)^{\diamond_n 1}$.

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- $U^{0,n} = 1 = \left(B^{H,n}\right)^{\diamond_n 0}$ and $U^{1,n} = \left(B^{H,n}\right)^{\diamond_n 1}$.
- But $U_2^{2,n} \;=\; 2B_{\frac{1}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} \;\neq\; B_{\frac{2}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} \;=\; \left( B_{\frac{2}{n}}^{H,n} \right)^{\diamond_n 2}$.

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- $U^{0,n} = 1 = \left(B^{H,n}\right)^{\diamond_n 0}$ and $U^{1,n} = \left(B^{H,n}\right)^{\diamond_n 1}$.
- But $U_2^{2,n} = 2B_{\frac{1}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} \neq B_{\frac{2}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} = \left( B_{\frac{2}{n}}^{H,n} \right)^{\diamond_n 2}$.
- The discrete Wick powers are not the solutions for (8) if $k \geq 2$.

**Theorem**

*Suppose*

1. $\lim_{n\to\infty} a_{n,k} = a_k$ *exists for all* $k \in \mathbb{N}$.
2. *There exists a* $C \in \mathbb{R}_+$, *so that* $|a_{n,k}| \leq C^k$ *for all* $n, k \in \mathbb{N}$.

*Define* $\widetilde{U}_t^{k,n} := U_{\lfloor nt \rfloor}^{k,n}$ *as the piecewise constant interpolation of (8). Then the sequence of processes* $\sum_{k=0}^{n} \frac{a_{n,k}}{k!} \widetilde{U}^{k,n}$ *converges weakly to the Wick power series* $\sum_{k=0}^{\infty} \frac{a_k}{k!} \left(B^H\right)^{\diamond k}$ *in the Skorokhod space* $D([0,1],\mathbb{R})$.

## Example (Geometric fractional Brownian motion)

$$\exp^{\diamond_n}\left(B_t^{H,n}\right) := \sum_{k=0}^{\lfloor nt \rfloor} \frac{1}{k!}\left(B_t^{H,n}\right)^{\diamond_n k} \xrightarrow{d} \exp^{\diamond}\left(B^H\right),$$

$$\widetilde{S}^n := \sum_{k=0}^{n} \frac{1}{k!}\widetilde{U}^{k,n} \xrightarrow{d} \exp^{\diamond}\left(B^H\right).$$

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$$S_l^n = S_{l-1}^n + S_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), \qquad S_0^n = 1 \tag{9}$$

for $l = 1, \ldots, n$, where $S_l^n = \widetilde{S}_{\frac{l}{n}}^n$.

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for $l = 1, \ldots, n$, where $S_l^n = \widetilde{S}_{\frac{l}{n}}^n$.

Hence, the piecewise constant interpolation of (9) converges weakly to the solution of the fractional Doléans-Dade equation.

## Example (Wick-sine and Wick-cosine)

The piecewise constant interpolation of

$$\begin{aligned} X_l^n &= X_{l-1}^n + Y_{l-1}^n \diamond_n \left( B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), X_0^n = 0, \\ Y_l^n &= Y_{l-1}^n - X_{l-1}^n \diamond_n \left( B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), Y_0^n = 1, \end{aligned}$$

converges weakly to the solution of the linear system

$$\begin{aligned} dX_t &= Y_t d^\diamond B_t^H \qquad X_0 = 0, \\ dY_t &= -X_t d^\diamond B_t^H \qquad Y_0 = 1, \end{aligned}$$

the process $\left(\sin^\diamond\left(B_t^H\right), \cos^\diamond\left(B_t^H\right)\right)^T$.

## Example (Linear SDE with drift)

Suppose $\mu, s_0 \in \mathbb{R}$, $\sigma > 0$. Then $\widetilde{S}_t^n := S_{\lfloor nt \rfloor}^n$, where $S^n$ is the solution of the Wick difference equation

$$S_l^n = \left(1 + \frac{\mu}{n}\right) S_{l-1}^n + \sigma S_{l-1}^n \diamond_n \left( B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), S_0^n = s_0, \quad (10)$$

converges weakly to the solution of the *linear SDE with drift*

$$dS_t = \mu S_t dt + \sigma S_t d^{\diamond} B_t^H, \qquad S_0 = s_0 \quad (11)$$

in the Skorokhod space $D([0,1], \mathbb{R})$.

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- This was conjectured by Bender and Elliott in their study of the discrete Wick-fractional Black-Scholes market.

Denote

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## Proposition

$$\frac{1}{k!} U^{k,n}_l = \sum_{\substack{C \subseteq \{1,\dots,l\} \\ |C|=k}} \left( \sum_{\substack{m : C \to \{1,\dots,l\} \\ \text{injective}}} \prod_{p \in C} \left( b^n_{\frac{m}{n},p} - b^n_{\frac{m-1}{n},p} \right) \right) \Psi^n_C, \quad (12)$$

$$\frac{1}{k!} \left( B^{H,n}_{\frac{l}{n}} \right)^{\diamond_n k} = \sum_{\substack{C \subseteq \{1,\dots,l\} \\ |C|=k}} b^n_{\frac{l}{n},C} \Psi^n_C, \quad (13)$$

## Proposition

*For all* $t \in [0,1]$ *and* $i \in \{1, \ldots, \lfloor nt \rfloor\}$,

$$b_{t,i}^{n} \leq 2c_H n^{-(1-H)}.$$

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## Theorem (**Nieminen** 2004)

$$\mathbf{E}\left[B^{H,n}_t B^{H,n}_s\right] \longrightarrow \mathbf{E}\left[B^H_t B^H_s\right].$$

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$$\mathbf{E}\left[B_t^{H,n} B_s^{H,n}\right] \longrightarrow \mathbf{E}\left[B_t^H B_s^H\right].$$

## Proposition

$$\lim_{n\to\infty} \mathbf{E}\left[\left(\left(B_t^{H,n}\right)^{\diamond_n N} - \left(B_s^{H,n}\right)^{\diamond_n N}\right)^2\right] = \mathbf{E}\left[\left(\left(B_t^H\right)^{\diamond N} - \left(B_s^H\right)^{\diamond N}\right)^2\right].$$

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2. Tightness of the sequences of processes

$$\left(\sum_{k=0}^{n} \frac{a_{n,k}}{k!} \left(B^{H,n}\right)^{\diamond_n k}\right)_{n\in\mathbb{N}}, \left(\sum_{k=0}^{n} \frac{a_{n,k}}{k!} \widetilde{U}^{k,n}\right)_{n\in\mathbb{N}}.$$

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**Proposition (Discrete Hermite recursion)**

$$(B_t^{H,n})^{\diamond_n (N+1)} = B_t^{H,n}(B_t^{H,n})^{\diamond_n N} - N\mathbf{E}\left[(B_t^{H,n})^2\right](B_t^{H,n})^{\diamond_n (N-1)} + R(B_t^{H,n}, N), \quad (14)$$

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$$R(B_t^{H,n}, N) = N! \sum_{\substack{C\subseteq\{1,\dots\lfloor nt\rfloor\} \\ |C|=N-1}} b_{t,C}^n \Psi_C^n \sum_{i\in C} \left(b_{t,i}^n\right)^2, \qquad (15)$$

$$\mathbf{E}\left[\left(R(B_t^{H,n}, N)\right)^2\right] \leq 16 c_H^4 N! N^3 n^{-(4-4H)}. \qquad (16)$$

### Theorem

*For all* $N \in \mathbb{N}$,

$$\left(1, B^{H,n}, \dots, (B^{H,n})^{\diamond_n N}\right) \xrightarrow{fd} \left(1, B^H, \dots, (B^H)^{\diamond N}\right). \tag{17}$$

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## An idea of proof: Billingsley Theorem 4.2.

$$\forall m \in \mathbb{N} \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \xrightarrow{fd} \sum_{k=0}^{m} \frac{a_k}{k!} (B_t^H)^{\diamond k} \text{ as } n \to \infty,$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}\left[ \left| \sum_{k=0}^{n} \frac{a_{n,k}}{k!} \left(B_t^{H,n}\right)^{\diamond_n k} - \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \right| \wedge 1 \right] = 0,$$

$$\sum_{k=0}^{m} \frac{a_k}{k!} (B_t^H)^{\diamond k} \xrightarrow{fd} \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k} \qquad \text{as } m \to \infty.$$

# Tightness

## Theorem (a variant of Billingsley Theorem 15.6)

*Suppose for the random elements $Y^n$ in the Skorokhod space $D([0,1],\mathbb{R})$ and $\sum_{k=0}^{\infty} \frac{a_k}{k!} \left(B^H\right)^{\diamond k}$ in $C([0,1],\mathbb{R})$,*

$$Y^n \xrightarrow{fd} \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(B^H\right)^{\diamond k},$$

$$s \le t \ , \ \mathbf{E}\left[(Y^n_t - Y^n_s)^2\right] \le L \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H},$$

*where $L > 0$ is a constant. Then $Y^n$ converges weakly to $\sum_{k=0}^{\infty} \frac{a_k}{k!} \left(B^H\right)^{\diamond k}$ in $D([0,1],\mathbb{R})$.*

## Lemma

*Let* $(X, \langle \cdot, \cdot \rangle)$ *be a real inner product space,* $N \geq 1$,

$$\|x\|^{2N} + \|y\|^{2N} - 2\left(\langle x, y\rangle\right)^N \leq 2^{N+1}\left(\|x\| + \|y\|\right)^{2(N-1)} \|x - y\|^2 .$$

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## Lemma

$$\frac{1}{N!}\mathbf{E}\left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\right)^2\right] \leq 8^N \left|\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n}\right|^{2H} .$$

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## An idea of proof.

$$\frac{1}{N!}\mathbf{E}\left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\right)^2\right]$$
$$\leq \mathbf{E}\left[(B_t^{H,n})^2\right]^N + \mathbf{E}\left[(B_s^{H,n})^2\right]^N - 2\mathbf{E}\left[(B_t^{H,n})(B_s^{H,n})\right]^N$$

# Generalizations

## Theorem

*Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function. Then $I_H(f)^n$ converges weakly to $I_H(f) = \int_0^{\cdot} f(s)dB_s^H$ in the Skorokhod space, where*

$$I_H(f)_t^n := \sum_{i=1}^{\lfloor nt \rfloor} f\left(\frac{i-1}{n}\right)\left(B^{H,n}_{\frac{i}{n}} - B^{H,n}_{\frac{i-1}{n}}\right)$$

$$= \sum_{i=1}^{n} \xi_i^n \left(\sum_{j=i}^{\lfloor nt \rfloor} f\left(\frac{j-1}{n}\right)\left(b^n_{\frac{j}{n},i} - b^n_{\frac{j-1}{n},i}\right)\right)$$

*is the discrete Wiener integral.*

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