Recent progress concerning the diaphony of generalised van der Corput sequences in arbitrary base b

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Motivation

- Precise study of various discrepancy measures for generalised van der Corput sequences led to more general results in higher bases.
- Like always the main goal is "to try to construct explicitly good point sets - this is the end of the story then".

Irregularities of Distribution

Let $X = (x_n)_{n \geq 1}$ be a one-dimensional infinite sequence. For $N \geq 1$ and for an interval $I := [\alpha, \beta[$, where α, $\beta \in [0, 1]$

$$A(I, N, X)$$

gives the number of indices $n \leq N$, for which $x_n \in I$. We call

$$E(I, N, X) = A(I, N, X) - \lambda(I)N.$$

the discrepancy function of the interval I.

Diaphony

Let $X = (x_n)_{n\geq 1}$ be a one-dimensional infinite sequence. The diaphony F of the first N points of X is defined by (Zinterhof, 1976)

$$F(N,X) := \left(2 \cdot \sum_{m=1}^{\infty} \frac{1}{m^2} \left| \sum_{n=1}^{N} \exp^{2i\pi m x_n} \right|^2 \right)^{1/2}.$$

Or in terms of the discrepancy function:

$$F^2(N,X) = 2\pi^2 \int_0^1 \int_0^1 E^2([\alpha,\beta[\,;N;X) d\alpha d\beta.$$

Generalised van der Corput sequence

Definition

Given an integer $n \geq 1$ in b-adic representation $\sum_{j=0}^{\infty} a_j(n) b^j$ and a permutation $\sigma \in \mathfrak{S}_b$, then the generalised van der Corput sequence S_b^σ in fixed base b is defined by

$$S_b^\sigma(n) = \sum_{j=0}^{\infty} \frac{\sigma(a_j(n))}{b^{j+1}}.$$

The generalised van der Corput sequence is a low discrepancy sequence!

Example

Let $b = 5$,
$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 & 4 \end{pmatrix}$.

n	$\ldots 5^2 5^1 5^0$	x_n
0	000	$\frac{\sigma(0)}{5} = 0$
1	001	$\frac{\sigma(1)}{5} = \frac{3}{5}$
2	002	$\frac{\sigma(2)}{5} = \frac{2}{5}$
3	003	$\frac{\sigma(3)}{5} = \frac{1}{5}$
4	004	$\frac{\sigma(4)}{5} = \frac{4}{5}$
5	010	$\frac{\sigma(0)}{5} + \frac{\sigma(1)}{25} = \frac{3}{25}$
6	011	$\frac{\sigma(1)}{5} + \frac{\sigma(1)}{25} = \frac{3}{5} + \frac{3}{25}$
7	012	$\frac{\sigma(2)}{5} + \frac{\sigma(1)}{25} = \frac{2}{5} + \frac{3}{25}$
8	013	$\frac{\sigma(3)}{5} + \frac{\sigma(1)}{25} = \frac{1}{5} + \frac{3}{25}$
9	014	$\frac{\sigma(4)}{5} + \frac{\sigma(1)}{25} = \frac{4}{5} + \frac{3}{25}$
$\vdots$	$\vdots$	$\vdots$

Analysis of the diaphony I

For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \dots, \sigma(b-1)/b)$. For $h \in \{0, 1, \dots, b-1\}$ and $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$, where $k \in \{1, \dots, b\}$ we define

Definition

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A([0, \frac{h}{b}); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([\frac{h}{b}, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where for a sequence $X = (x_n)_{n \geq 1}$ we denote by $A(I; k; X)$ the number of indices $1 \leq n \leq k$ such that $x_n \in I$.

Analysis of the diaphony II

In 1993 Chaix and Faure introduced a new class of functions based on the basic $\varphi^{\sigma}_{b,h}$:

Definition

$$\chi^{\sigma}_{b} := \frac{1}{2} \sum_{h \neq h'} (\varphi^{\sigma}_{b,h'} - \varphi^{\sigma}_{b,h})^2.$$

Note that χ^{σ}_{b} is continuous and piecewise quadratic on the intervals $[\frac{k}{b}, \frac{k+1}{b}]$.

Analysis of the diaphony III

Moreover we have the following:

Chaix/Faure, 1993, Propriete 3.5

For each interval $[\frac{k}{b}, \frac{k+1}{b}]$ the parabolic arcs of χ_b^σ are translated versions of the parabola $y = b^2(b^2 - 1)x^2/12$.

Remark

This means that it suffices to know the values of χ_b^σ at the b-adic positions $\frac{k}{b}$ since for every interval $[\frac{k}{b}, \frac{k+1}{b}]$ χ_b^σ is of the form $Ax^2 + Bx + C$ with A only depending on the base b.

Graph of a χ-function

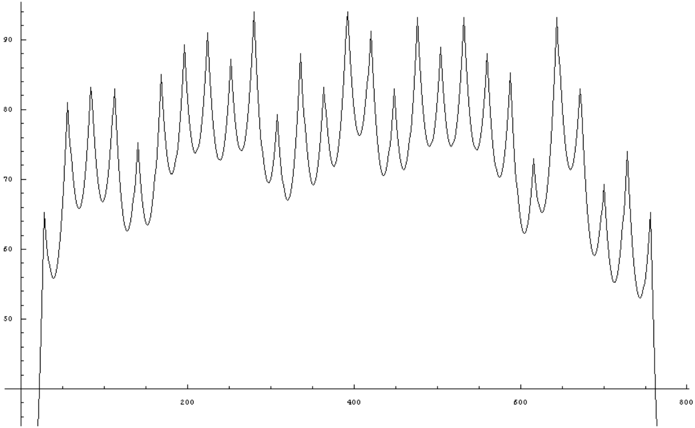

Figure: Graph of a χ-function in base 28

Important Theorems I

Chaix/Faure, 1993, Theorem 4.2

For all $N \geq 1$, we have

$$F^2(N, S_b^\sigma) = 4\pi^2 \sum_{j=1}^{\infty} \chi_b^\sigma(Nb^{-j}))/b^2.$$

Important Theorems II

Chaix/Faure, 1993, Theorem 4.10

Let

$$\gamma_b^\sigma = \inf_{n\geq 1} \sup_{x\in\mathbb{R}} \left(\sum_{j=1}^{n} \chi_b^\sigma(x/b^j)/n \right),$$

then

$$f(S_b^\sigma) := \limsup_{N\to\infty} (F^2(N,X)/\log N) = 4\pi^2\gamma_b^\sigma/(b^2\log b).$$

Crucial Lemma

In 2005 H. Faure proved the following lemma

Faure, 2005, Lemma 3

For any permutation σ in arbitrary base b it holds that

$$\chi_b^{\sigma} \leq \chi_b^{I}.$$

Analysis of the proof I

The proof is based on the following observations. First recall that

$$\chi_b^\sigma = \frac{1}{2} \sum_{h \neq h'} (\varphi_{b,h'}^\sigma - \varphi_{b,h}^\sigma)^2.$$

For arbitrary $h \neq h'$,

$$\varphi_{b,h'}^\sigma(\frac{k}{b}) - \varphi_{b,h}^\sigma(\frac{k}{b}) = E\left(\left[\frac{h}{b}, \frac{h'}{b}\right), k, Z_b^\sigma\right).$$

Hence proving the lemma amounts to proving that for fixed k

$$\sum_{h \neq h'} \left(E\left(\left[\frac{h}{b}, \frac{h'}{b}\right), k, Z_b^\sigma\right)\right)^2 \leq \sum_{h \neq h'} \left(E\left(\left[\frac{h}{b}, \frac{h'}{b}\right), k, Z_b^I\right)\right)^2.$$

Analysis of the proof II

One can show that we get or fixed $d := h - h'$ with

$$E\left(\left[\frac{h}{b}, \frac{h'}{b}\right), k, Z_b^\sigma\right) = \delta_{h,h'}^\sigma - \frac{dk}{b},$$

and $\delta_{h,h'}^\sigma := A([\frac{h}{b}, \frac{h'}{b}; k; Z_b^\sigma))$

$$\sum_{l(h,h')=d} (\delta_{h,h'}^\sigma - \frac{dk}{b})^2 = \sum_{l(h,h')=d} (\delta_{h,h'}^\sigma)^2 - \frac{d^2k^2}{b^2}$$

for any σ.

Analysis of the proof III

Hence to prove the lemma, we only need to compare

$$\sum_{l(h,h')=d} (\delta^{\sigma}_{h,h'})^2 \text{ and } \sum_{l(h,h')=d} (\delta^{I}_{h,h'})^2 \tag{1}$$

with the condition

$$\sum_{l(h,h')=d} (\delta^{\sigma}_{h,h'}) = \sum_{l(h,h')=d} (\delta^{I}_{h,h'}) = kd. \tag{2}$$

Crucial idea

We actually sum over all pairs (h, h') with $h \neq h'$. We can now take a more systematic look at these sums and rewrite them as $(b-1) \times b$ matrices, where

- the i in the i-th row denotes b times the length of the interval $[\frac{h}{b}, \frac{h'}{b}[$,
- the j in the j-th column stands for the left bound of the interval
- and at position (i, j) we put the number of the first k points that actually lie in the interval.

Illustration

For a permutation σ we denote the matrix corresponding to the first k points with $\mathcal{M}_k^\sigma$. Moreover we define $\mathcal{N}(\mathcal{M}_k^\sigma)$ as the sum over the squares of all entries of the matrix.

$$
\begin{array}{r c l}
 & \begin{array}{ccccccc} 0 & \cdot & \cdot & j & \cdot & \cdot & b-1 \end{array} & \\
\begin{array}{r} d=1 \\ \cdot \\ \cdot \\ d=i \\ \cdot \\ \cdot \\ d=b-1 \end{array} &
\left(\begin{array}{ccccccc}
x_{1,0} & \cdot & \cdot & \cdot & \cdot & \cdot & x_{b-1,0} \\
\cdot & \cdot & & & & \cdot & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & & x_{i,j} & & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & & & \cdot & \cdot \\
x_{1,b-1} & \cdot & \cdot & \cdot & \cdot & \cdot & x_{b-1,b-1}
\end{array}\right) &
\begin{array}{l} \sum_{row} = k \\ \cdot \\ \cdot \\ \sum_{row} = ik \\ \cdot \\ \cdot \\ \sum_{row} = (b-1)k \end{array}
\end{array}
$$

Illustration

If a new point is added at position j, in other words if we turn from k to $k+1$ points, then the following triangle of 1s is added to the matrix $\mathcal{M}_k^\sigma$ of step k where the 1 in the first row is placed at $x_{1,j}$:

$$\begin{pmatrix}
 & & & 1 & & & & & \\
 & & . & . & & & & & \\
 & . & & . & & & & & \\
1 & . & . & 1 & & & & & \\
. & & & . & & & & & 1 \\
. & & & . & & & & . & . \\
. & & & . & & & . & & . \\
1 & . & . & 1 & 0 & 1 & . & . & 1
\end{pmatrix}$$

Example

For example for let $b = 6$, $k = 2$. On the left side the first two points are at positions 0, 1, whereas on the right side the points are at positions 0, 3.

$$\begin{pmatrix} 1 & 1 & & & & \\ 2 & 1 & & & & 1 \\ 2 & 1 & & & 1 & 2 \\ 2 & 1 & & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & & & 1 & & \\ 1 & & 1 & 1 & & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 \end{pmatrix}$$

In the left matrix we have 10 times 1 and 10 times 2 whereas in the right matrix we have 18 times 1 and only 6 times 2.

Main questions

(1) How can these sums of squares of entries of the matrices be minimized?

(2) Can we determine the number of classes of permutations with different behaviour?

(3) Is it possible to determine a class of permutations that minimize these sums?

Results I

For $kd \equiv \gamma \mod b$, with $\gamma \in \{0, \dots, b-1\}$, let

$$L_b^{k,d} := (b-\gamma)(\lfloor \frac{kd}{b} \rfloor)^2 + \gamma(\lfloor \frac{kd}{b} + 1 \rfloor)^2.$$

P. & S. , 2010, Theorem 1

For arbitrary base b and arbitrary $\sigma \in \mathfrak{S}_b$ and for each k, $1 \leq k \leq b$, it holds that

$$\chi_b^\sigma(\frac{k}{b}) \geq L_b^k$$

with $L_b^k = \frac{1}{2} \sum_{d=1}^{b-1} \left(L_b^{k,d} - \frac{d^2 k^2}{b} \right)$.

Some further Notations

For two integers a and b let us consider their corresponding triangles of 1s. We will denote the number of fields where these triangles intersect with

$$F(a, b)$$

For example for $base = 6$, $a = 2$, $b = 4$ we have $F(2,4) = 7$:

$$\begin{pmatrix} & & 1 & & 1 & \\ & 1 & 1 & 1 & 1 & \\ 1 & 1 & 2 & 1 & 1 & \\ 1 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}$$

Results II

P. & S. , 2010, Theorem 2

Let $\mathcal{Z}_b^{k,\sigma_1} \neq \mathcal{Z}_b^{k,\sigma_2}$. Then

$$\mathcal{N}(\mathcal{M}_k^{\sigma_1}) = \mathcal{N}(\mathcal{M}_k^{\sigma_2})$$

if and only if there exists a bijective map
$f : \{(a, b) : a, b \in \mathcal{Z}_b^{k,\sigma_1}\} \to \{(a, b) : a, b \in \mathcal{Z}_b^{k,\sigma_2}\}$ such that

$$F(a, b) = F(f(a, b)).$$

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