Polynomial Lattice Rules: Old and New Results

Friedrich Pillichshammer[1]

Linz/Austria

[1]Supported by the Austrian Science Foundation (FWF), Project S9609.

Multivariate integration – QMC

We want to approximate an integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

using a QMC rule

$$Q_{N,s}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

where $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1} \in [0,1)^s$.

Multivariate integration – QMC

We want to approximate an integral

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

using a QMC rule

$$Q_{N,s}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

where $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1} \in [0,1)^s$.

Point sets with good distribution properties yield a small error.

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \leq V(f) D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \le V(f) D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

For $\mathbf{z} \in (0,1]^s$ let

$$\Delta(\mathbf{z}) := \frac{\#\{0 \le n < N : \mathbf{x}_n \in [\mathbf{0}, \mathbf{z})\}}{N} - \lambda_s([\mathbf{0}, \mathbf{z})).$$

Multivariate integration – QMC

Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \le V(f) D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}),$$

where

- $V(f)$ is a measure for the variation of f;
- D_N^* is the star discrepancy of the nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$.

For $\mathbf{z} \in (0,1]^s$ let

$$\Delta(\mathbf{z}) := \frac{\#\{0 \le n < N \,:\, \mathbf{x}_n \in [\mathbf{0}, \mathbf{z})\}}{N} - \lambda_s([\mathbf{0}, \mathbf{z})).$$

Definition (star discrepancy)

$$D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) := \sup_{\mathbf{z} \in (0,1]^s} |\Delta(\mathbf{z})|$$

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Bilyk, Lacey, Vagharshakyan 2008)

$$D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \text{ where } \kappa_s \geq \frac{s-1}{2}$$

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Bilyk, Lacey, Vagharshakyan 2008)

$$D_N^*(\{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \text{ where } \kappa_s \geq \frac{s-1}{2} + \delta_s$$

and $0 < \delta_s < \frac{1}{2}$.

There are several methods to construct point sets with "low" star discrepancy:

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Bilyk, Lacey, Vagharshakyan 2008)

$$D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \text{ where } \kappa_s \geq \frac{s-1}{2} + \delta_s$$

and $0 < \delta_s < \frac{1}{2}$.

There are several methods to construct point sets with "low" star discrepancy:

- van der Corput, Halton-sequence, Hammersley point set;

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Bilyk, Lacey, Vagharshakyan 2008)

$$D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \text{ where } \kappa_s \geq \frac{s-1}{2} + \delta_s$$

and $0 < \delta_s < \frac{1}{2}$.

There are several methods to construct point sets with "low" star discrepancy:

- van der Corput, Halton-sequence, Hammersley point set;
- lattice point sets (Hlawka, Korobov, Niederreiter, Sloan, Joe, Hickernell, Larcher, ...);

Multivariate integration – QMC

Lower bound on D_N^* (Roth 1954; Bilyk, Lacey, Vagharshakyan 2008)

$$D_N^*(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) \geq c_s \frac{(\log N)^{\kappa_s}}{N} \text{ where } \kappa_s \geq \frac{s-1}{2} + \delta_s$$

and $0 < \delta_s < \frac{1}{2}$.

There are several methods to construct point sets with "low" star discrepancy:

- van der Corput, Halton-sequence, Hammersley point set;
- lattice point sets (Hlawka, Korobov, Niederreiter, Sloan, Joe, Hickernell, Larcher, ...);
- (t, m, s)-nets in base b (Niederreiter, Sobol', Faure, Niederreiter-Xing, Chen-Skriganov, ...).

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t,m,s)-**net in base** b if every b-adic elementary interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t, m, s)-**net in base** b if every b-adic elementary interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- small **quality parameter** $0 \leq t \leq m$ implies good distribution properties;

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t, m, s)-**net in base** b if every b-adic elementary interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- small **quality parameter** $0 \le t \le m$ implies good distribution properties;
- any $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is a (m, m, s)-net in base b;

(t, m, s)-net in base b

Definition (Niederreiter 1987)

A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called (t,m,s)-**net in base** b if every b-adic elementary interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

- small **quality parameter** $0 \le t \le m$ implies good distribution properties;
- any $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is a (m,m,s)-net in base b;
- **Niederreiter:** if $\mathcal{P}$ is a (t,m,s)-net in base b, $N = b^m$, then

$$D_N^*(\mathcal{P}) = O_{s,b}\left(b^t \frac{(\log N)^{s-1}}{N}\right).$$

(t, m, s)-nets: an example

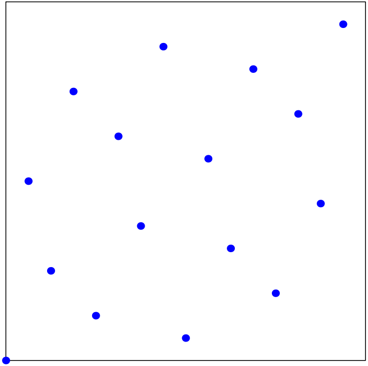


Figure: $(0, 4, 2)$-net in base 2.

(t, m, s)-nets: an example

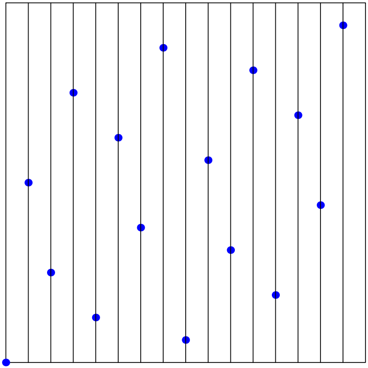


Figure: $(0, 4, 2)$-net in base 2.

(t, m, s)-nets: an example

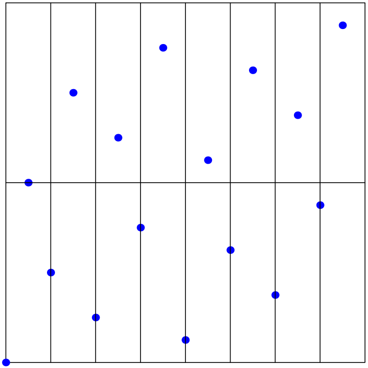


Figure: $(0, 4, 2)$-net in base 2.

(t, m, s)-nets: an example

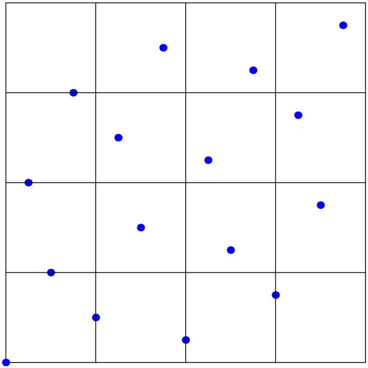


Figure: $(0, 4, 2)$-net in base 2.

(t, m, s)-nets: an example

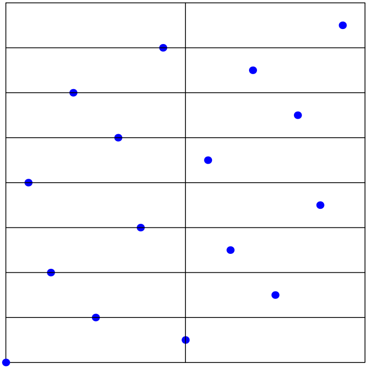


Figure: $(0, 4, 2)$-net in base 2.

(t, m, s)-nets: an example

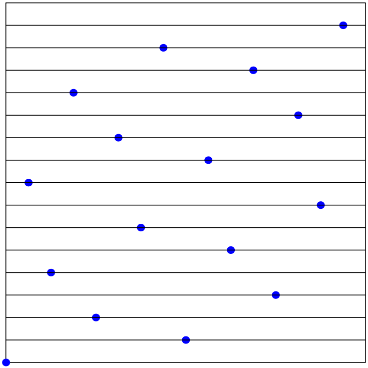


Figure: $(0, 4, 2)$-net in base 2.

Lattice point sets

Lattice point set

For $N \geq 2$ and $\mathbf{g} \in \mathbb{Z}^s$ let

$$\mathbf{x}_n = \left\{ \frac{n}{N} \mathbf{g} \right\} \quad \text{for} \quad 0 \leq n < N.$$

A QMC rule using a lattice point set is called a **lattice rule**.

Lattice point sets

Lattice point set

For $N \geq 2$ and $\mathbf{g} \in \mathbb{Z}^s$ let

$$\mathbf{x}_n = \left\{ \frac{n}{N} \mathbf{g} \right\} \quad \text{for} \quad 0 \leq n < N.$$

A QMC rule using a lattice point set is called a **lattice rule**.

Polynomial lattice point sets are in their over all structure very similar to (classical) lattice point sets.

lattice point set	**polynomial lattice point set**
based on number theoretic concepts	based on algebraic methods (polynomial arithmetic)

Polynomial lattice point sets

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.

Polynomial lattice point sets

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.
Let $\mathbb{Z}_b((x^{-1}))$ the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell},$$

where $w \in \mathbb{Z}$ and all $t_\ell \in \mathbb{Z}_b$.

Polynomial lattice point sets

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.
Let $\mathbb{Z}_b((x^{-1}))$ the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell},$$

where $w \in \mathbb{Z}$ and all $t_\ell \in \mathbb{Z}_b$.
For $n \in \mathbb{N}$ let

$$\nu_n : \mathbb{Z}_b((x^{-1})) \to [0,1)$$

$$\nu_n\left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}\right) = \sum_{\ell=\max(1,w)}^{n} t_\ell b^{-\ell}.$$

Polynomial lattice point sets

Let $b \in \mathbb{P}$ and let $\mathbb{Z}_b$ be the finite field with b elements.
Let $\mathbb{Z}_b((x^{-1}))$ the field of **formal Laurent series** over $\mathbb{Z}_b$ (b prime). Its elements are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell},$$

where $w \in \mathbb{Z}$ and all $t_\ell \in \mathbb{Z}_b$.
For $n \in \mathbb{N}$ let

$$\nu_n : \mathbb{Z}_b((x^{-1})) \to [0,1)$$

$$\nu_n \left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \right) = \sum_{\ell=\max(1,w)}^{n} t_\ell b^{-\ell}.$$

Let $G_{b,m} := \{h \in \mathbb{Z}_b[x] \,:\, \deg(h) < m\}$. We have $|G_{b,m}| = b^m$.

Polynomial lattice point sets

Definition (Niederreiter 1992)

For $s, m \in \mathbb{N}$, choose $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$. Then $\mathcal{P}(\mathbf{q}, p)$ is the point set consisting of the b^m points

$$\mathbf{x}_h = \nu_m\left(\frac{h(x)}{p(x)}\mathbf{q}(x)\right) \quad \text{where } h \in G_{b,m}.$$

Polynomial lattice point sets

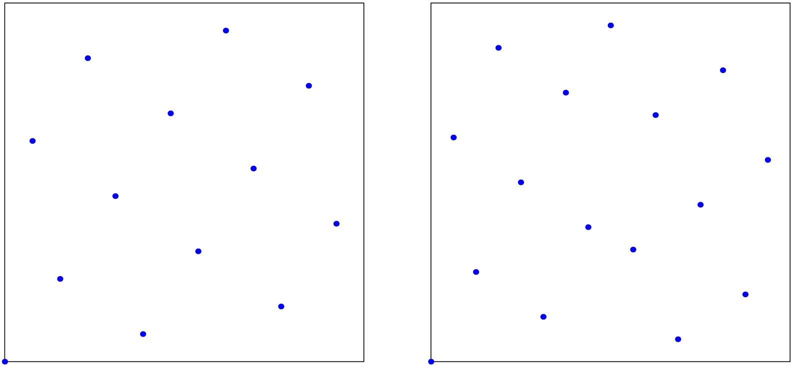
Definition (Niederreiter 1992)

For $s, m \in \mathbb{N}$, choose $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
Then $\mathcal{P}(\mathbf{q}, p)$ is the point set consisting of the b^m points

$$\mathbf{x}_h = \nu_m\left(\frac{h(x)}{p(x)}\mathbf{q}(x)\right) \quad \text{where } h \in G_{b,m}.$$

$\mathcal{P}(\mathbf{q}, p)$ is called a **polynomial lattice point set** and a QMC rule using $\mathcal{P}(\mathbf{q}, p)$ is called a **polynomial lattice rule**.

(Polynomial) lattice point sets: examples



left: $N = 13$, $\mathbf{g} = (1, 8)$
right: $p(x) = x^4 + x^2 + 1$, $\mathbf{q} = (1, x^3)$ in $\mathbb{Z}_2[x]$

Polynomial lattice point sets: examples

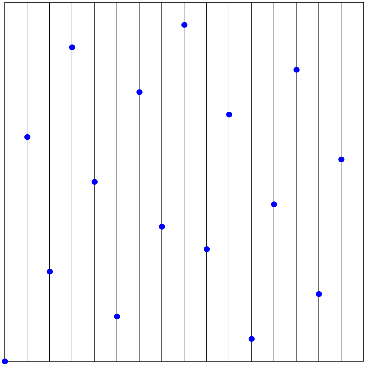


Figure: $\mathcal{P}((1, x^3), x^4 + x^2 + 1)$ as $(0, 4, 2)$-net in base 2.

Polynomial lattice point set: example

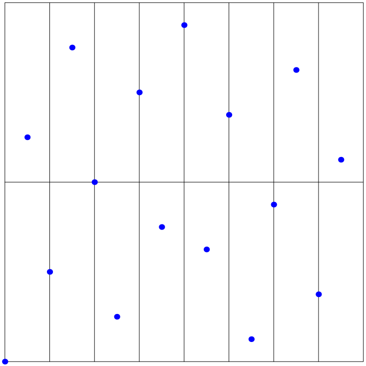


Figure: $\mathcal{P}((1, x^3), x^4 + x^2 + 1)$ as $(0, 4, 2)$-net in base 2.

Polynomial lattice point set: example

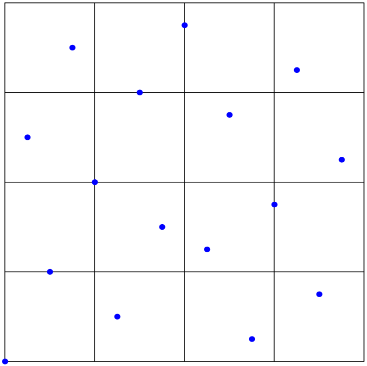


Figure: $\mathcal{P}((1, x^3), x^4 + x^2 + 1)$ as $(0, 4, 2)$-net in base 2.

Polynomial lattice point set: example

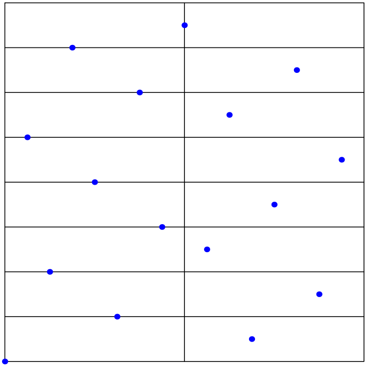


Figure: $\mathcal{P}((1, x^3), x^4 + x^2 + 1)$ as $(0, 4, 2)$-net in base 2.

Polynomial lattice point set: example

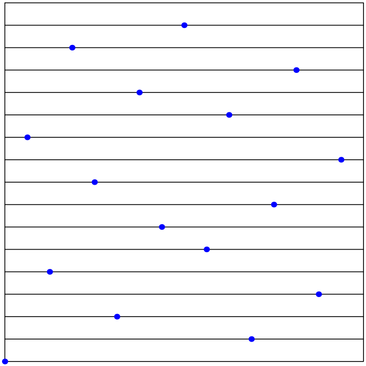
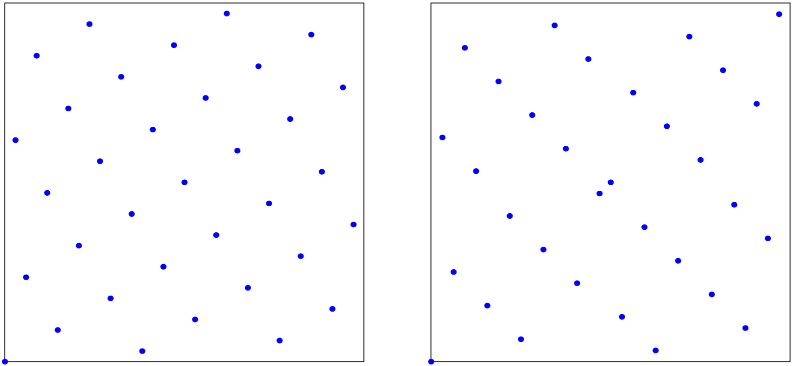


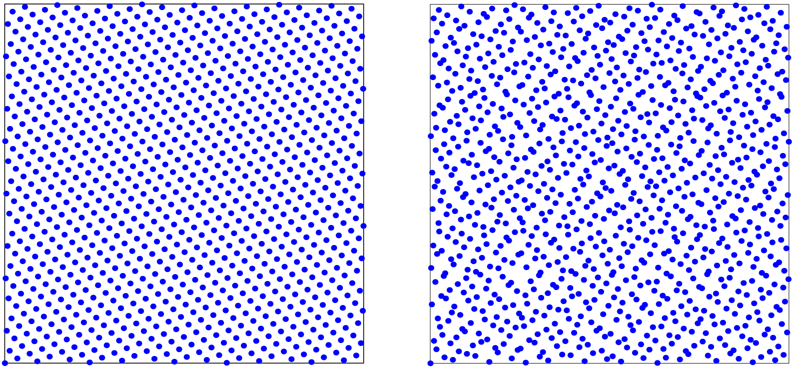
Figure: $\mathcal{P}((1, x^3), x^4 + x^2 + 1)$ as $(0, 4, 2)$-net in base 2.

(Polynomial) lattice point sets: examples



left: $N = 34$, $\mathbf{g} = (1, 21)$
right: $p(x) = x^5 + x$, $\mathbf{q} = (1, x^4 + x^2 + 1)$ in $\mathbb{Z}_2[x]$

(Polynomial) lattice point sets: examples



left: $N = 987$, $\mathbf{g} = (1, 610)$
right: $p(x) = x^{10} + x^8 + x^4 + x^2 + 1$, $\mathbf{q} = (1, x^9 + x^5 + x)$ in $\mathbb{Z}_2[x]$

The dual lattice of (polynomial) lattice point sets

For a lattice point set $\mathbf{x}_n = \left\{\frac{n}{N}\mathbf{g}\right\}$, $0 \le n < N$, where $\mathbf{g} \in \mathbb{Z}^s$ the **dual lattice** is

$$\mathcal{L}_{\mathbf{g},N} = \{\mathbf{h} \in \mathbb{Z}^s : \mathbf{h} \cdot \mathbf{g} \equiv 0 \pmod{N}\}.$$

The dual lattice of (polynomial) lattice point sets

For a lattice point set $\mathbf{x}_n = \left\{\frac{n}{N}\mathbf{g}\right\}$, $0 \le n < N$, where $\mathbf{g} \in \mathbb{Z}^s$ the **dual lattice** is

$$\mathcal{L}_{\mathbf{g},N} = \{\mathbf{h} \in \mathbb{Z}^s : \mathbf{h} \cdot \mathbf{g} \equiv 0 \pmod{N}\}.$$

Definition

The **dual net** of $\mathcal{P}(\mathbf{q}, p)$ with $p \in \mathbb{Z}_b[x]$, $\deg(p) = m$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$ is

$$\mathcal{D}_{\mathbf{q},p} = \{\mathbf{k} \in G_{b,m}^s : \mathbf{k} \cdot \mathbf{q} \equiv 0 \pmod{p}\}.$$

Walsh functions

- For $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_i \in \{0, \ldots, b-1\}$ the kth **Walsh function**

  $$_b\mathrm{wal}_k : [0,1) \to \mathbb{C}$$

  is defined by

  $$_b\mathrm{wal}_k(x) := \exp(2\pi \mathtt{i}(\xi_1 \kappa_0 + \cdots + \xi_{a+1}\kappa_a)/b),$$

  where $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ with infinitely many digits $\xi_i \neq b-1$.

Walsh functions

- For $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_i \in \{0, \ldots, b-1\}$ the kth **Walsh function**

$$_b\mathrm{wal}_k : [0,1) \to \mathbb{C}$$

is defined by

$$_b\mathrm{wal}_k(x) := \exp(2\pi \mathtt{i}(\xi_1 \kappa_0 + \cdots + \xi_{a+1}\kappa_a)/b),$$

where $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ with infinitely many digits $\xi_i \neq b-1$.

- For $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let

$$_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{i=1}^{s} {}_b\mathrm{wal}_{k_i}(x_i).$$

Walsh functions

- For $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_i \in \{0, \ldots, b-1\}$ the kth **Walsh function**

$$_b\mathrm{wal}_k : [0,1) \to \mathbb{C}$$

is defined by

$$_b\mathrm{wal}_k(x) := \exp(2\pi \mathtt{i}(\xi_1\kappa_0 + \cdots + \xi_{a+1}\kappa_a)/b),$$

where $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ with infinitely many digits $\xi_i \neq b-1$.

- For $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ let

$$_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{i=1}^{s} {_b\mathrm{wal}_{k_i}}(x_i).$$

$\{ {_b\mathrm{wal}_{\mathbf{k}}} : \mathbf{k} \in \mathbb{N}_0^s \}$ is a complete orthonormal system in $L_2([0,1]^s)$.

Walsh functions

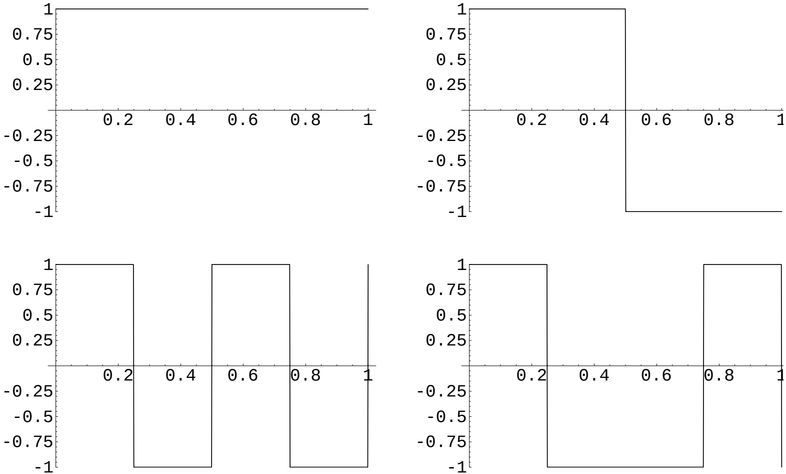


Figure: ${}_2\mathrm{wal}(x)$ for $k = 0, 1, 2, 3$.

The dual net of polynomial lattice point sets

Important property of lattice point sets:

$$\sum_{n=0}^{N-1} \exp\left(2\pi \mathtt{i}\{n\mathbf{g}/N\} \cdot \mathbf{k}\right) = \begin{cases} N & \text{if } \mathbf{k} \in \mathcal{L}_{\mathbf{g},N}, \\ 0 & \text{otherwise.} \end{cases}$$

The dual net of polynomial lattice point sets

Important property of lattice point sets:

$$\sum_{n=0}^{N-1} \exp\left(2\pi \mathtt{i} \{n\mathbf{g}/N\} \cdot \mathbf{k}\right) = \begin{cases} N & \text{if } \mathbf{k} \in \mathcal{L}_{\mathbf{g},N}, \\ 0 & \text{otherwise.} \end{cases}$$

For the analysis of the worst-case integration error it is most convenient to consider **one-periodic functions**.

The dual net of polynomial lattice point sets

Important property of lattice point sets:

$$\sum_{n=0}^{N-1} \exp\left(2\pi \mathtt{i} \{n\mathbf{g}/N\} \cdot \mathbf{k}\right) = \begin{cases} N & \text{if } \mathbf{k} \in \mathcal{L}_{\mathbf{g},N}, \\ 0 & \text{otherwise.} \end{cases}$$

For the analysis of the worst-case integration error it is most convenient to consider **one-periodic functions**.

Important property polynomial lattice point sets:

$$\sum_{\mathbf{x} \in \mathcal{P}(\mathbf{q},p)} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \begin{cases} b^m & \text{if } \mathbf{k} \in \mathcal{D}_{\mathbf{q},p}, \\ 0 & \text{otherwise.} \end{cases}$$

The dual net of polynomial lattice point sets

Important property of lattice point sets:

$$\sum_{n=0}^{N-1} \exp\left(2\pi \mathtt{i}\{n\mathbf{g}/N\} \cdot \mathbf{k}\right) = \begin{cases} N & \text{if } \mathbf{k} \in \mathcal{L}_{\mathbf{g},N}, \\ 0 & \text{otherwise.} \end{cases}$$

For the analysis of the worst-case integration error it is most convenient to consider **one-periodic functions**.

Important property polynomial lattice point sets:

$$\sum_{\mathbf{x} \in \mathcal{P}(\mathbf{q},p)} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \begin{cases} b^m & \text{if } \mathbf{k} \in \mathcal{D}_{\mathbf{q},p}, \\ 0 & \text{otherwise.} \end{cases}$$

For the analysis of the worst-case integration error it is most convenient to consider **Walsh series**.

Quality measures

For $p \in \mathbb{Z}_b[x]$ and $\mathbf{q} \in \mathbb{Z}_b[x]^s$ define

$$\rho(\mathbf{q}, p) = s - 1 + \min_{\mathbf{h} \in \mathcal{D}_{\mathbf{q},p} \setminus \{\mathbf{0}\}} \sum_{i=1}^{s} \deg(h_i)$$

Quality measures

For $p \in \mathbb{Z}_b[x]$ and $\mathbf{q} \in \mathbb{Z}_b[x]^s$ define

$$\rho(\mathbf{q}, p) = s - 1 + \min_{\mathbf{h} \in \mathcal{D}_{\mathbf{q},p} \setminus \{\mathbf{0}\}} \sum_{i=1}^{s} \deg(h_i)$$

$$R_b(\mathbf{q}, p) = \sum_{\mathbf{h} \in \mathcal{D}_{\mathbf{q},p} \setminus \{\mathbf{0}\}} \prod_{i=1}^{s} r_b(h_i),$$

where for $h \in G_{b,m}$

$$r_b(h) := \begin{cases} 1 & \text{if } h = 0, \\ \frac{1}{b^{r+1} \sin^2(\pi \kappa_r / b)} & \text{if } h = \kappa_0 + \kappa_1 b + \cdots + \kappa_r x^r,\ \kappa_r \neq 0. \end{cases}$$

Quality measures

Theorem (Niederreiter 1992)

$\mathcal{P}(\mathbf{q}, p)$ is a (t, m, s)-net in base b with $m = \deg(p)$,

$$t = m - \rho(\mathbf{q}, p)$$

and

$$D^*_{b^m}(\mathcal{P}(\mathbf{q}, p)) \le \frac{s}{b^m} + R_b(\mathbf{q}, p).$$

Existence results

Theorem (Larcher, Lauss, Niederreiter, Schmid 1996; Niederreiter 1992; Dick, Leobacher, P. 2005)

Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.

- If p is irreducible, then there exists $\mathbf{q} \in G_{b,m}^s$ such that

$$t \leq (s-1)\log_b m - (s-2) - \log_b \frac{(s-1)!}{(b-1)^{s-1}}.$$

Existence results

Theorem (Larcher, Lauss, Niederreiter, Schmid 1996; Niederreiter 1992; Dick, Leobacher, P. 2005)

Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.

- If p is irreducible, then there exists $\mathbf{q} \in G_{b,m}^s$ such that

$$t \leq (s-1)\log_b m - (s-2) - \log_b \frac{(s-1)!}{(b-1)^{s-1}}.$$

Hence $D_{b^m}^*(\mathcal{P}(\mathbf{q}, p)) = O_{s,b}\left(\frac{m^{2s-2}}{b^m}\right)$.

Existence results

Theorem (Larcher, Lauss, Niederreiter, Schmid 1996; Niederreiter 1992; Dick, Leobacher, P. 2005)

Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.

- If p is irreducible, then there exists $\mathbf{q} \in G_{b,m}^s$ such that

  $$t \leq (s-1)\log_b m - (s-2) - \log_b \frac{(s-1)!}{(b-1)^{s-1}}.$$

  Hence $D_{b^m}^*(\mathcal{P}(\mathbf{q}, p)) = O_{s,b}\left(\frac{m^{2s-2}}{b^m}\right)$.

- For $0 \leq \varepsilon < 1$ there are more than $\varepsilon |G_{b,m}^s|$ vectors $\mathbf{q} \in G_{b,m}^s$ with

  $$D_{b^m}^*(\mathcal{P}(\mathbf{q}, p)) \leq \frac{s}{b^m} + R_b(\mathbf{q}, p) = O_{s,b,\varepsilon}\left(\frac{m^s}{b^m}\right).$$

Existence results

Theorem (Kritzer, P. 2010)

There exists $c_{s,b} > 0$ such that for any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and any $\mathbf{q} \in G_{b,m}^s$, $q_i \neq 0$, $1 \leq i \leq s$, we have

$$R_b(\mathbf{q}, p) \geq c_{s,b} b^{\deg(\delta_s)} \frac{(m - \deg(\delta_s))^s}{b^m},$$

where $\delta_s := \gcd(q_1, \ldots, q_s, p)$.

Existence results

Theorem (Kritzer, P. 2010)

There exists $c_{s,b} > 0$ such that for any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and any $\mathbf{q} \in G_{b,m}^s$, $q_i \neq 0$, $1 \leq i \leq s$, we have

$$R_b(\mathbf{q}, p) \geq c_{s,b} b^{\deg(\delta_s)} \frac{(m - \deg(\delta_s))^s}{b^m},$$

where $\delta_s := \gcd(q_1, \ldots, q_s, p)$.

Theorem (Larcher 1993)

For any $m \geq 2$ there exists $\mathbf{q} \in G_{b,m}^s$ with

$$D_{b^m}^*(\mathcal{P}(\mathbf{q}, x^m)) = O_{s,b}\left(\frac{m^{s-1} \log m}{b^m}\right).$$

CBC-construction

Sloan and Joe

Let $\mathcal{Z}_N := \{1, \ldots, N-1\}$ and let $S_N(\mathbf{g})$ be a quality measure for a lattice point set.

CBC-construction

Sloan and Joe

Let $\mathcal{Z}_N := \{1, \ldots, N-1\}$ and let $S_N(\mathbf{g})$ be a quality measure for a lattice point set.

1. Choose $g_1 = 1$.

CBC-construction

Sloan and Joe

Let $\mathcal{Z}_N := \{1, \ldots, N-1\}$ and let $S_N(\mathbf{g})$ be a quality measure for a lattice point set.

1. Choose $g_1 = 1$.
2. For $s > 1$ assume that $g_1, \ldots, g_{s-1} \in \mathcal{Z}_N$ are already known. Find $g_s \in \mathcal{Z}_N$ which minimizes

$$S_N((g_1, \ldots, g_{s-1}, z))$$

as a function of z.

CBC-construction

Sloan and Joe

Let $\mathcal{Z}_N := \{1, \ldots, N-1\}$ and let $S_N(\mathbf{g})$ be a quality measure for a lattice point set.

1. Choose $g_1 = 1$.
2. For $s > 1$ assume that $g_1, \ldots, g_{s-1} \in \mathcal{Z}_N$ are already known. Find $g_s \in \mathcal{Z}_N$ which minimizes

$$S_N((g_1, \ldots, g_{s-1}, z))$$

   as a function of z.

This principle also works in the polynomial case (with $G_{b,m}$ instead of $\mathcal{Z}_N$).

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ a sequence of weights in $\mathbb{R}^+$. Let $\mathcal{I}_s = \{1, \ldots, s\}$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ a sequence of weights in $\mathbb{R}^+$. Let $\mathcal{I}_s = \{1, \ldots, s\}$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

Weighted star discrepancy (Sloan and Woźniakowski 1998)

The **weighted star discrepancy** is given by

$$D^*_{N,\boldsymbol{\gamma}}(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) = \sup_{\mathbf{z} \in (0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} |\Delta((z_{\mathfrak{u}}, 1))|.$$

The weighted star discrepancy

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ a sequence of weights in $\mathbb{R}^+$. Let $\mathcal{I}_s = \{1, \ldots, s\}$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$.

Weighted star discrepancy (Sloan and Woźniakowski 1998)

The **weighted star discrepancy** is given by

$$D^*_{N,\boldsymbol{\gamma}}(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) = \sup_{\mathbf{z} \in (0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} |\Delta((z_{\mathfrak{u}}, 1))|.$$

Weighted Koksma-Hlawka inequality

$$|I_s(f) - Q_{N,s}(f)| \leq D^*_{N,\boldsymbol{\gamma}}(\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}) \|f\|_{s,\boldsymbol{\gamma}}.$$

The weighted star discrepancy

Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and let $\mathbf{q} \in G_{b,m}^s$.

$$D^*_{b^m,\boldsymbol{\gamma}}(\mathcal{P}(\mathbf{q},p)) \le \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{b^m}\right)^{|\mathfrak{u}|}\right) + R_{b,\boldsymbol{\gamma}}(\mathbf{q},p),$$

where

$$R_{b,\boldsymbol{\gamma}}(\mathbf{q},p) = \sum_{\mathbf{h} \in \mathcal{D}_{\mathbf{q},p} \setminus \{\mathbf{0}\}} \prod_{i=1}^{s} r_b(h_i,\gamma_i).$$

Here for $h \in G_{b,m}$ we put

$$r_b(h,\gamma) = \begin{cases} 1+\gamma & \text{if } h = 0, \\ \gamma r_b(h) & \text{if } h \ne 0. \end{cases}$$

$R_{b,\boldsymbol{\gamma}}(\mathbf{q},p)$ can be computed in $O(b^m s)$ operations.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\boldsymbol{\gamma} = (\gamma_i)_{i \geq 1}$.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.

The weighted star discrepancy – CBC-construction

CBC-algorithm

Given a prime b, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\gamma = (\gamma_i)_{i \geq 1}$.

1. Choose $q_1 = 1$.
2. For $s > 1$, assume we have already constructed $q_1, \ldots, q_{s-1} \in G_{b,m}^*$. Then find $q_s \in G_{b,m}^*$ which minimises the quantity

$$R_{b,\gamma}((q_1, \ldots, q_{s-1}, z), p)$$

as a function of z.

The weighted star discrepancy – CBC-construction

Theorem (Dick, Leobacher, P. 2005)

If $\mathbf{q} \in G_{b,m}^s$ is constructed with the CBC-algorithm, then

$$R_{b,\gamma}(\mathbf{q}, p) \leq \frac{1}{b^m - 1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(1 + m\frac{b^2 - 1}{3b}\right)\right),$$

The weighted star discrepancy – CBC-construction

Theorem (Dick, Leobacher, P. 2005)

If $\mathbf{q} \in G_{b,m}^s$ is constructed with the CBC-algorithm, then

$$R_{b,\boldsymbol{\gamma}}(\mathbf{q}, p) \leq \frac{1}{b^m - 1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(1 + m\frac{b^2 - 1}{3b}\right)\right),$$

Corollary

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $c_{\boldsymbol{\gamma},\delta} > 0$, such that

$$D_{b^m,\boldsymbol{\gamma}}^*(\mathcal{P}(\mathbf{q}, p)) \leq \frac{c_{\boldsymbol{\gamma},\delta}}{b^{m(1-\delta)}}.$$

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathcal{P}(\mathbf{q}^{(j)}, p^{(j)})$ with the CBC-algorithm.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathcal{P}(\mathbf{q}^{(j)}, p^{(j)})$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathcal{P}(\mathbf{q}^{(j)}, p^{(j)})$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

Theorem (Hinrichs, P., Schmid 2008)

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $C_{\boldsymbol{\gamma},\delta} > 0$, such that

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_N) \leq \frac{C_{\boldsymbol{\gamma},\delta}}{N^{1-\delta}}.$$

The weighted star discrepancy – strong tractability

- Let $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \leq m_1 < m_2 < \ldots < m_k$.
- For $1 \leq j \leq k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ with $\deg(p^{(j)}) = m_j$ and construct $\mathcal{P}(\mathbf{q}^{(j)}, p^{(j)})$ with the CBC-algorithm.
- Set $\mathcal{P}_N = \mathcal{P}(\mathbf{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\mathbf{q}^{(k)}, p^{(k)})$.

Theorem (Hinrichs, P., Schmid 2008)

If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $C_{\boldsymbol{\gamma},\delta} > 0$, such that

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_N) \leq \frac{C_{\boldsymbol{\gamma},\delta}}{N^{1-\delta}}.$$

The weighted star discrepancy is **strongly polynomial tractable** with ε-exponent equal to one.

CBC-construction – cost

- The cost for the CBC-algorithm is of $O(b^{2m}s^2)$ operations.

CBC-construction – cost

- The cost for the CBC-algorithm is of $O(b^{2m}s^2)$ operations.
- Nuyens and Cools (2006) introduced the Fast CBC-construction with a significan reduction of cost.

CBC-construction – cost

- The cost for the CBC-algorithm is of $O(b^{2m}s^2)$ operations.
- Nuyens and Cools (2006) introduced the Fast CBC-construction with a significan reduction of cost.
- Cost of the Fast CBC-algorithm $O(smb^m)$ with $O(b^m)$ memory space.

Integration of Walsh series

Let $\alpha > 1$ and let $\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ be the weighted Hilbert function space with reproducing kernel given by

$$K_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \rho_\alpha(\mathbf{k},\boldsymbol{\gamma})\, {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\, \overline{{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{y})},$$

where for $\mathbf{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ we put $\rho_\alpha(\mathbf{k},\boldsymbol{\gamma}) = \prod_{j=1}^s \rho_\alpha(k_j,\gamma_j)$ with

$$\rho_\alpha(k,\gamma) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\alpha v} & \text{if } b^v \leq k < b^{v+1} \text{ for } v \in \mathbb{N}_0. \end{cases}$$

Integration of Walsh series

Let $\alpha > 1$ and let $\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ be the weighted Hilbert function space with reproducing kernel given by

$$K_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \rho_\alpha(\mathbf{k},\boldsymbol{\gamma})\, {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{y})},$$

where for $\mathbf{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ we put $\rho_\alpha(\mathbf{k},\boldsymbol{\gamma}) = \prod_{j=1}^s \rho_\alpha(k_j,\gamma_j)$ with

$$\rho_\alpha(k,\gamma) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\alpha v} & \text{if } b^v \leq k < b^{v+1} \text{ for } v \in \mathbb{N}_0. \end{cases}$$

$$\|f\|_{\mathscr{H}_{\mathrm{wal},s,\alpha,\gamma}} = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \rho_\alpha(\mathbf{k},\boldsymbol{\gamma})^{-1} |\widehat{f}_{\mathrm{wal}}(\mathbf{k})|^2$$

where $\widehat{f}_{\mathrm{wal}}(\mathbf{k}) = \int_{[0,1]^s} f(\mathbf{x})\overline{{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x})}\,\mathrm{d}\mathbf{x}$.

Integration of Walsh series

Worst-case error of $\mathcal{P}(\mathbf{q}, p)$

$$e(\mathbf{q}, p) := \sup_{\|f\|_{\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}} \le 1} |I_s(f) - Q_{b^m,s}(f)|$$

Integration of Walsh series

Worst-case error of $\mathcal{P}(\mathbf{q}, p)$

$$
\begin{aligned}
e(\mathbf{q}, p) &:= \sup_{\|f\|_{\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}} \le 1} |I_s(f) - Q_{b^m,s}(f)| \\
e^2(\mathbf{q}, p) &= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ tru_m(\mathbf{k})(x) \in \mathcal{D}_{\mathbf{q},p}}} \rho_\alpha(\mathbf{k}, \gamma)
\end{aligned}
$$

where $tru_m(\mathbf{k}) :\equiv \mathbf{k} \pmod{b^m}$ and where

$$k = \kappa_0 + \kappa_1 b + \ldots + \kappa_{m-1} b^{m-1} \in \mathbb{N}_0$$

is identified with

$$k(x) = \kappa_0 + \kappa_1 x + \ldots + \kappa_{m-1} x^{m-1} \in \mathbb{Z}_b[x].$$

Integration of Walsh series

Theorem (Dick, Kuo, P., Sloan 2005; Kritzer, P. 2007)

For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ we can construct CBC $\mathbf{q} \in G_{b,m}^s$ such that (with $N = b^m$)

1. We have

$$e(\mathbf{q}, p) \leq \frac{c_{s,\alpha,\boldsymbol{\gamma},\delta}}{N^{\alpha/2-\delta}} \qquad \text{for all } 0 < \delta \leq \frac{\alpha-1}{2}.$$

Integration of Walsh series

Theorem (Dick, Kuo, P., Sloan 2005; Kritzer, P. 2007)

For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ we can construct CBC $\mathbf{q} \in G_{b,m}^s$ such that (with $N = b^m$)

1. We have
$$e(\mathbf{q}, p) \le \frac{c_{s,\alpha,\boldsymbol{\gamma},\delta}}{N^{\alpha/2-\delta}} \qquad \text{for all } \ 0 < \delta \le \tfrac{\alpha-1}{2}.$$

2. Suppose $\sum_{i=1}^{\infty} \gamma_i^{1/(\alpha-2\delta)} < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\delta} \le c_{\infty,\alpha,\boldsymbol{\gamma},\delta} < \infty$ and we have
$$e(\mathbf{q}, p) \le \frac{c_{\infty,\alpha,\boldsymbol{\gamma},\delta}}{N^{\alpha/2-\delta}}.$$

Extensible polynomial lattice point sets

Disadvantage

The CBC-algorithm and hence the vectors $\mathbf{q}$ depend on p and hence on $N = b^{\deg(p)}$. If one changes p, then one has to construct a new vector $\mathbf{q} \in \mathbb{Z}_b[x]^s$.

Extensible polynomial lattice point sets

Disadvantage

The CBC-algorithm and hence the vectors $\mathbf{q}$ depend on p and hence on $N = b^{\deg(p)}$. If one changes p, then one has to construct a new vector $\mathbf{q} \in \mathbb{Z}_b[x]^s$.

For $p \in \mathbb{Z}_b[x]$ with $m = \deg(p) \geq 1$, let Y_p be the set of all p-adic polynomials

$$\sum_{n=0}^{\infty} a_n p^n \quad \text{with} \quad \deg(a_n) < m.$$

Then

$$Y_p/(p^n) = G_{b,nm}.$$

Extensible polynomial lattice point sets

Let $\mathbf{Q} \in Y_p^s$ and for $n \in \mathbb{N}$ let $\mathbf{q}_n \equiv \mathbf{Q} \pmod{p^n}$. Then

$$\mathcal{P}(\mathbf{q}_1, p) \subseteq \mathcal{P}(\mathbf{q}_2, p^2) \subseteq \mathcal{P}(\mathbf{q}_3, p^3) \subseteq \ldots.$$

Extensible polynomial lattice point sets

Let $\mathbf{Q} \in Y_p^s$ and for $n \in \mathbb{N}$ let $\mathbf{q}_n \equiv \mathbf{Q} \pmod{p^n}$. Then

$$\mathcal{P}(\mathbf{q}_1, p) \subseteq \mathcal{P}(\mathbf{q}_2, p^2) \subseteq \mathcal{P}(\mathbf{q}_3, p^3) \subseteq \ldots .$$

Extensible polynomial lattice point set

$$\mathcal{P}(\mathbf{Q}, p) := \mathcal{P}(\mathbf{q}_1, p) \cup \mathcal{P}(\mathbf{q}_2, p^2) \cup \mathcal{P}(\mathbf{q}_3, p^3) \cup \ldots .$$

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

- Start with a good vector $\mathbf{q}_1 \in G_{b,m}^s$.

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

- Start with a good vector $\mathbf{q}_1 \in G_{b,m}^s$.
- $\mathbf{q}_2 = \mathbf{q}_1 + p\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_2, p^2)$.

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

- Start with a good vector $\mathbf{q}_1 \in G_{b,m}^s$.
- $\mathbf{q}_2 = \mathbf{q}_1 + p\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_2, p^2)$.
- $\mathbf{q}_3 = \mathbf{q}_2 + p^2\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_3, p^3)$.

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

- Start with a good vector $\mathbf{q}_1 \in G_{b,m}^s$.
- $\mathbf{q}_2 = \mathbf{q}_1 + p\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_2, p^2)$.
- $\mathbf{q}_3 = \mathbf{q}_2 + p^2\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_3, p^3)$.
- This way we can find step by step a good generating vector for all $p, p^2, p^3, \ldots$.

Construction of extensible polynomial lattice point sets

For $\mathcal{P}(\mathbf{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\mathbf{Q}$ are important.

- Start with a good vector $\mathbf{q}_1 \in G_{b,m}^s$.
- $\mathbf{q}_2 = \mathbf{q}_1 + p\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_2, p^2)$.
- $\mathbf{q}_3 = \mathbf{q}_2 + p^2\mathbf{q}$ where $\mathbf{q} \in G_{b,m}^s$ is choosen to minimize $e^2(\mathbf{q}_3, p^3)$.
- This way we can find step by step a good generating vector for all $p, p^2, p^3, \ldots$.
- Advantage: one does not have to stop at some a priori fixed p^v.

Construction of extensible polynomial lattice point sets

Algorithm

Let $p \in \mathbb{Z}_b[x]$ be monic and irreducible with $\deg(p) = m$.

1. Find $\mathbf{q}_1 := \mathbf{q}$ by minimizing $e^2(\mathbf{q}, p)$ over all $\mathbf{q} \in G_{b,m}^s$.
2. For $n = 2, 3, \ldots$ find $\mathbf{q}_n := \mathbf{q}_{n-1} + p^{n-1}\mathbf{q}$ by minimizing $e^2(\mathbf{q}_{n-1} + p^{n-1}\mathbf{q}, p^n)$ over all $\mathbf{q} \in G_{b,m}^s$.

Construction of extensible polynomial lattice point sets

Algorithm

Let $p \in \mathbb{Z}_b[x]$ be monic and irreducible with $\deg(p) = m$.

1. Find $\mathbf{q}_1 := \mathbf{q}$ by minimizing $e^2(\mathbf{q}, p)$ over all $\mathbf{q} \in G_{b,m}^s$.
2. For $n = 2, 3, \ldots$ find $\mathbf{q}_n := \mathbf{q}_{n-1} + p^{n-1}\mathbf{q}$ by minimizing $e^2(\mathbf{q}_{n-1} + p^{n-1}\mathbf{q}, p^n)$ over all $\mathbf{q} \in G_{b,m}^s$.

Theorem (Niederreiter, P. 2009)

If $\mathbf{q}_n \in G_{b,m}^s$ is constructed according to the above algorithm, then

$$e^2(\mathbf{q}_n, p^n) \leq \frac{c_{s,b,\gamma,\alpha}}{b^{nm}}.$$

Integration in Sobolev spaces

Similar results hold for the mean square worst-case error of **digitally shifted polynomial lattices** for integration in the Sobolev space with reproducing kernel

$$K(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{s} \left(1 + \gamma_i B_1(x_i) B_1(y_i) + \frac{\gamma_i}{2} B_2(|x_i - y_i|)\right).$$

Integration in Sobolev spaces

Similar results hold for the mean square worst-case error of **digitally shifted polynomial lattices** for integration in the Sobolev space with reproducing kernel

$$K(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{s} \left(1 + \gamma_i B_1(x_i) B_1(y_i) + \frac{\gamma_i}{2} B_2(|x_i - y_i|)\right).$$

Theorem (Dick, Kuo, P., Sloan 2005)

$$\widehat{e}^2(\mathbf{q}, p) \leq \frac{c_{s,b,\boldsymbol{\gamma},\varepsilon}}{N^{2-\varepsilon}} \quad \text{for } \varepsilon > 0$$

where $N = b^m$.

Integration in Sobolev spaces

Theorem (Baldeaux, Dick 2010)

Scrambled polynomial lattice point sets yield (randomized setting)

$$\mathbb{E}\left[|I_s(f) - Q_{N,s}(f)|^2\right] \leq \frac{c_{s,b,\boldsymbol{\gamma},\varepsilon}}{N^{3-\varepsilon}} \quad \text{for } \varepsilon > 0$$

where $N = b^m$.

Integration in Sobolev spaces

Sobolev space with reproducing kernel

$$K(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{s} \left(1 + \gamma_i B_1(x_i) B_1(y_i) + \frac{\gamma_i^2}{4} B_2(x_i) B_2(y_i) - \frac{\gamma_i^2}{24} B_4(|x_i - y_i|)\right).$$

Integration in Sobolev spaces

Sobolev space with reproducing kernel

$$K(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{s}\left(1+\gamma_i B_1(x_i)B_1(y_i)+\frac{\gamma_i^2}{4}B_2(x_i)B_2(y_i)-\frac{\gamma_i^2}{24}B_4(|x_i-y_i|)\right).$$

Digitally shifted polynomial lattices over $\mathbb{Z}_2$ + tent transformation $\phi(x)=1-|2x-1|$ (Hickernell 2000) yield

Theorem (Cristea, Dick, Leobacher, P. 2008)

$$\widehat{e}_{\phi}^2(\mathbf{q},p)\le\frac{c_{s,\boldsymbol{\gamma},\varepsilon}}{N^{4-\varepsilon}}\quad\text{for } \varepsilon>0$$

where $N=2^m$.

Higher order polynomial lattice rules

For

$$k = \kappa_1 b^{a_1 - 1} + \kappa_2 b^{a_2 - 1} + \cdots + \kappa_v b^{a_v - 1},$$

where $1 \leq a_v < \cdots < a_1$, $v \geq 1$ and $\kappa_1, \ldots, \kappa_v \in \{1, \ldots, b-1\}$, define

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(v,\alpha)}, \qquad \alpha \geq 1.$$

Higher order polynomial lattice rules

For

$$k = \kappa_1 b^{a_1 - 1} + \kappa_2 b^{a_2 - 1} + \cdots + \kappa_v b^{a_v - 1},$$

where $1 \leq a_v < \cdots < a_1$, $v \geq 1$ and $\kappa_1, \ldots, \kappa_v \in \{1, \ldots, b-1\}$, define

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(v,\alpha)}, \qquad \alpha \geq 1.$$

For $k \in \mathbb{N}_0$ and $\gamma > 0$, define

$$r_\alpha(k, \gamma) := \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\mu_\alpha(k)} & \text{otherwise.} \end{cases}$$

Higher order polynomial lattice rules

For

$$k = \kappa_1 b^{a_1 - 1} + \kappa_2 b^{a_2 - 1} + \cdots + \kappa_v b^{a_v - 1},$$

where $1 \le a_v < \cdots < a_1$, $v \ge 1$ and $\kappa_1, \ldots, \kappa_v \in \{1, \ldots, b-1\}$, define

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(v,\alpha)}, \qquad \alpha \ge 1.$$

For $k \in \mathbb{N}_0$ and $\gamma > 0$, define

$$r_\alpha(k, \gamma) := \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\mu_\alpha(k)} & \text{otherwise.} \end{cases}$$

For $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, set

$$r_\alpha(\mathbf{k}, \boldsymbol{\gamma}) := \prod_{i=1}^{s} r_\alpha(k_i, \gamma_i).$$

Higher order polynomial lattice rules

Let $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}} \subseteq L_2([0,1]^s)$ be the space consisting of all Walsh series $f = \sum_{\mathbf{k} \in \mathbb{N}_0^s} \widehat{f}_{\mathrm{wal}}(\mathbf{k})\, {}_b\mathrm{wal}_{\mathbf{k}}$ for which

$$\|f\|_{\mathscr{W}_{\alpha,s,\gamma}} := \sup_{\mathbf{k} \in \mathbb{N}_0^s} \frac{|\widehat{f}_{\mathrm{wal}}(\mathbf{k})|}{r_\alpha(\mathbf{k}, \gamma)} < \infty$$

Higher order polynomial lattice rules

Let $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}} \subseteq L_2([0,1]^s)$ be the space consisting of all Walsh series $f = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \widehat{f}_{\mathrm{wal}}(\mathbf{k})\, {}_b\mathrm{wal}_{\mathbf{k}}$ for which

$$\|f\|_{\mathscr{W}_{\alpha,s,\gamma}} := \sup_{\mathbf{k}\in\mathbb{N}_0^s} \frac{|\widehat{f}_{\mathrm{wal}}(\mathbf{k})|}{r_\alpha(\mathbf{k},\gamma)} < \infty$$

Theorem (Dick 2008)

Let $\alpha \geq 2$ and let $f : [0,1]^s \to \mathbb{R}$ such that all mixed partial derivatives up to order α in each variable are square integrable, then $f \in \mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}$.

Higher order polynomial lattice rules

Definition (Dick, P. 2006)

For $s, m, n \in \mathbb{N}$, $m \leq n$, choose $p \in \mathbb{Z}_b[x]$, with $\deg(p) = n$ and let $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
Then $\mathcal{P}_{m,n}(\mathbf{q}, p)$ is the point set consisting of the b^m points

$$\mathbf{x}_h = \nu_n \left(\frac{h(x)}{p(x)} \mathbf{q}(x) \right) \quad \text{where } h \in G_{b,m}.$$

Higher order polynomial lattice rules

Definition (Dick, P. 2006)

For $s, m, n \in \mathbb{N}$, $m \leq n$, choose $p \in \mathbb{Z}_b[x]$, with $\deg(p) = n$ and let $\mathbf{q} \in \mathbb{Z}_b[x]^s$.
Then $\mathcal{P}_{m,n}(\mathbf{q}, p)$ is the point set consisting of the b^m points

$$\mathbf{x}_h = \nu_n\left(\frac{h(x)}{p(x)}\mathbf{q}(x)\right) \quad \text{where } h \in G_{b,m}.$$

Remark

For $m = n$ we have $\mathcal{P}_{m,m}(\mathbf{q}, p) = \mathcal{P}(\mathbf{q}, p)$.

Higher order polynomial lattice rules

Definition

The **dual net** of $\mathcal{P}_{m,n}(\mathbf{q}, p)$ with $p \in \mathbb{Z}_b[x]$, $\deg(p) = n$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$ is given by

$$\mathcal{D}_{\mathbf{q},p} = \{\mathbf{k} \in G_{b,n}^s : \mathbf{k} \cdot \mathbf{q} \equiv u \,(\mathrm{mod}\ p) \text{ with } \deg(u) < n - m\}.$$

Higher order polynomial lattice rules

Definition

The **dual net** of $\mathcal{P}_{m,n}(\mathbf{q}, p)$ with $p \in \mathbb{Z}_b[x]$, $\deg(p) = n$, and $\mathbf{q} \in \mathbb{Z}_b[x]^s$ is given by

$$\mathcal{D}_{\mathbf{q},p} = \{\mathbf{k} \in G_{b,n}^s \,:\, \mathbf{k} \cdot \mathbf{q} \equiv u \pmod{p} \text{ with } \deg(u) < n - m\}.$$

Theorem (Dick, P. 2007)

For $\alpha \geq 2$ the worst-case error for integration in $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}$ using $\mathcal{P}_{m,n}(\mathbf{q}, p)$ is

$$e_\alpha^2(\mathbf{q}, p) = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ tru_n(\mathbf{k})(x) \in \mathcal{D}_{\mathbf{q},p}}} r_\alpha(\mathbf{k}, \boldsymbol{\gamma}).$$

Higher order polynomial lattice rules

Theorem (Baldeaux, Dick, Greslehner and P. 2010)

For any irreducible $p \in \mathbb{Z}_b[x]$ with $\deg(p) = n$ we can construct CBC $\mathbf{q} \in G_{b,n}^s$ such that

$$e_\alpha(\mathbf{q}, p) \leq \frac{c_{s,\alpha,\gamma,\tau}}{b^{\min(\tau m, n)}} \text{ for all } 1 \leq \tau < \alpha.$$

Higher order polynomial lattice rules

Theorem (Baldeaux, Dick, Greslehner and P. 2010)

For any irreducible $p \in \mathbb{Z}_b[x]$ with $\deg(p) = n$ we can construct CBC $\mathbf{q} \in G_{b,n}^s$ such that

$$e_\alpha(\mathbf{q}, p) \le \frac{c_{s,\alpha,\gamma,\tau}}{b^{\min(\tau m, n)}} \text{ for all } 1 \le \tau < \alpha.$$

- Choosing n large we obtain a convergence order of $N^{-\alpha+\varepsilon}$ for $\varepsilon > 0$ where $N = b^m$.
- This convergence rate is essentially best possible (Sharygin).

Higher order polynomial lattice rules

Corollary

Suppose $\sum_{i=1}^{\infty} \gamma_i^{1/\tau} < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\tau} \leq c_{\infty,\alpha,\boldsymbol{\gamma},\tau} < \infty$ and we have

$$e_\alpha(\mathbf{q}^*, p) \leq \frac{c_{\infty,\alpha,\boldsymbol{\gamma},\tau}}{b^{\min(\tau m, n)}} \quad \text{for all } 1 \leq \tau < \alpha.$$

Higher order polynomial lattice rules

Corollary

Suppose $\sum_{i=1}^{\infty} \gamma_i^{1/\tau} < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\tau} \leq c_{\infty,\alpha,\boldsymbol{\gamma},\tau} < \infty$ and we have

$$e_\alpha(\mathbf{q}^*, p) \leq \frac{c_{\infty,\alpha,\boldsymbol{\gamma},\tau}}{b^{\min(\tau m, n)}} \quad \text{for all } 1 \leq \tau < \alpha.$$

Optimal convergence rates for a **range** of smoothness parameters.

Higher order polynomial lattice rules

Corollary

Suppose $\sum_{i=1}^{\infty} \gamma_i^{1/\tau} < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\tau} \le c_{\infty,\alpha,\boldsymbol{\gamma},\tau} < \infty$ and we have

$$e_\alpha(\mathbf{q}^*, p) \le \frac{c_{\infty,\alpha,\boldsymbol{\gamma},\tau}}{b^{\min(\tau m, n)}} \quad \text{for all } 1 \le \tau < \alpha.$$

Optimal convergence rates for a **range** of smoothness parameters.
Sieve principle: Let $A, B \subseteq X$, $|X| < \infty$. If $|A|, |B| > |X|/2$, then $|A \cap B| > 0$.

Higher order polynomial lattice rules

Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and set $n = \beta m$. Let $p \in \mathbb{Z}_b[x]$, $\deg(p) = \beta m$.

Higher order polynomial lattice rules

Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and set $n = \beta m$. Let $p \in \mathbb{Z}_b[x]$, $\deg(p) = \beta m$.

Sieve Algorithm

1. Find $\lfloor (1-\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\mathbf{q}$ in $G_{b,\beta m}^s$ which satisfy

$$e_2(\mathbf{q}, p) \leq \frac{c_{s,b,\gamma,s,m,\beta,\alpha=2,\tau_2}}{b^{\tau_2 m}} \quad \text{for all } 1 \leq \tau_2 < 2,$$

and label this set $\mathcal{T}_2$.

Higher order polynomial lattice rules

Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and set $n = \beta m$. Let $p \in \mathbb{Z}_b[x]$, $\deg(p) = \beta m$.

Sieve Algorithm

1. Find $\lfloor (1-\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\mathbf{q}$ in $G_{b,\beta m}^s$ which satisfy

$$e_2(\mathbf{q}, p) \leq \frac{C_{s,b,\gamma,s,m,\beta,\alpha=2,\tau_2}}{b^{\tau_2 m}} \quad \text{for all } 1 \leq \tau_2 < 2,$$

   and label this set $\mathcal{T}_2$.

2. Then find $\lfloor (1-2\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\mathbf{q}$ in $\mathcal{T}_2$ which satisfy

$$e_3(\mathbf{q}, p) \leq \frac{C_{s,b,\gamma,s,m,\beta,\alpha=3,\tau_3}}{b^{\tau_3 m}} \quad \text{for all } 1 \leq \tau_3 < 3$$

   and label this set $\mathcal{T}_3$.

Higher order polynomial lattice rules

Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and set $n = \beta m$. Let $p \in \mathbb{Z}_b[x]$, $\deg(p) = \beta m$.

Sieve Algorithm

1. Find $\lfloor (1-\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\mathbf{q}$ in $G_{b,\beta m}^s$ which satisfy
$$e_2(\mathbf{q}, p) \leq \frac{C_{s,b,\gamma,s,m,\beta,\alpha=2,\tau_2}}{b^{\tau_2 m}} \quad \text{for all } 1 \leq \tau_2 < 2,$$
and label this set $\mathcal{T}_2$.
2. Then find $\lfloor (1-2\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\mathbf{q}$ in $\mathcal{T}_2$ which satisfy
$$e_3(\mathbf{q}, p) \leq \frac{C_{s,b,\gamma,s,m,\beta,\alpha=3,\tau_3}}{b^{\tau_3 m}} \quad \text{for all } 1 \leq \tau_3 < 3$$
and label this set $\mathcal{T}_3$.
3. In the same way we proceed to construct the sets $\mathcal{T}_4, \ldots, \mathcal{T}_\beta$. Select $\mathbf{q}^*$ to be any vector from $\mathcal{T}_\beta$.

Higher order polynomial lattice rules

Combination of the Sieve Algorithm and the CBC approach yields:

Theorem (Baldeaux, Dick, Greslehner, P. 2010)

Let $s, m, \beta \in \mathbb{N}$, $\beta \geq 2$, then one can construct a vector $\mathbf{q} \in G_{b,\beta m}^s$ such that

$$e_\alpha(\mathbf{q}, p) \leq \frac{c_{s,b,\alpha,\beta,\gamma,\tau_\alpha}}{b^{\tau_\alpha m}} \text{ for all } 1 \leq \tau_\alpha < \alpha$$

and for all $2 \leq \alpha \leq \beta$.

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES
tent transformation	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES
tent transformation	YES	YES
optimal conv. for smooth periodic fcts.	YES	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES
tent transformation	YES	YES
optimal conv. for smooth periodic fcts.	YES	YES
... for smooth non periodic fcts. + tractability	NO	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES
tent transformation	YES	YES
optimal conv. for smooth periodic fcts.	YES	YES
... for smooth non periodic fcts. + tractability	NO	YES
scrambling	NO	YES

Comparison lattice rules – polynomial lattice rules

lattice rules – polynomial lattice rules

property	LR	PLR
discrepancy bounds	YES	YES
tractability properties	YES	YES
CBC construction	YES	YES
fast CBC construction	YES	YES
extensible rules	YES	YES
tent transformation	YES	YES
optimal conv. for smooth periodic fcts.	YES	YES
... for smooth non periodic fcts. + tractability	NO	YES
scrambling	NO	YES
exponential convergence	YES	???