The optimality of Euler–type algorithms for approximation of stochastic differential equations with discontinuous coefficients

Paweł Przybyłowicz

AGH University of Science and Technology, Cracow, Poland
Department of Applied Mathematics
Przybyl83@gmail.com

MCQMC 2010, 15–20 August 2010, Warsaw, Poland

Outline:

- Problem formulation (basic notions and definitions),
- Known results,
- Main results (sketch of the construction of adaptive algorithms).

- P. Przybyłowicz, *On the optimality of Euler–type algorithms for approximation of stochastic differential equations with discontinuous coefficients*, manuscript in prep.,
- B. Kacewicz, P. Przybyłowicz, *Optimal adaptive solution of initial-value problems with unknown singularities*, J. Complexity 24 (4), 455–476, 2008,
- P. Przybyłowicz, *Linear information for the approximation of the Itô integrals*, Numer. Algorithms 52, 677-699, 2009,
- P. Przybyłowicz, *Adaptive Itô–Taylor algorithm can optimally approximate the Itô integrals of singular functions*, J. Comp. and Appl. Math 235, 203–217, 2010,
- G. W. Wasilkowski, H.Woźniakowski, *On the complexity of stochastic integration*, Math. Comp. 70, 685-698, 2001,

- L. Plaskota, G.W.Wasilkowski, *Adaption allows efficient integration of functions with unknown singularities*, Numer. Math. 102 , 123–144, 2005,
- N. Hofmann, T. Müller–Gronbach, K. Ritter, *The Optimal Discretization of Stochastic Differential Equations*, J. Complexity 17, 117–153, 2001,
- T. Müller–Gronbach, *Optimal pointwise approximation of SDEs based on Brownian motion at discrete points*, Ann. Appl. Probab. 14(4), 1605–1642, 2004,
- L. Yan, *The Euler scheme with irregular coefficients*, Ann. Probab. 30, no. 3, 1172–1194, 2002.

Problem formulation:

$T > 0$, $\sigma_1, \sigma_2 : [0, T] \to \mathbb{R}$, $a, b : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} dX(t) = \sigma_1(t)a(X(t))dt + \sigma_2(t)b(X(t))dW(t), & t \in [0, T], \\ X(0) = \eta, \end{cases} \tag{1}$$

σ_1, σ_2– regular or singular, a, b– at least Lipschitz functions,

Goal

Optimal approximation of $X(T)$.

Classes of functions:

$\sigma_1, \sigma_2 \in \mathcal{F}^{\varrho}_{\mathrm{reg}} \vee \mathcal{F}^{\varrho}_{\mathrm{sng},p}$, $\varrho \in (0,1]$, $p \in \mathbb{N}_+ \cup \{0\}$,

- $\mathcal{F}^{\varrho}_{\mathrm{reg}}$ – class of regular Hölder functions,
- $\mathcal{F}^{\varrho}_{\mathrm{sng},p}$ – class of piecewise Hölder continuous functions with jumps in discontinuity points,

$$\Delta_g^q := g(s_g^{q+}) - g(s_g^{q-}), \quad q = 1, .., p,$$

$a, b \in \mathcal{D}^1 \vee \mathcal{D}^2$,

- $\mathcal{D}^1 \subset C(\mathbb{R})$: $f-$ bounded in zero Lipschitz functions,
- $\mathcal{D}^2 \subset \mathcal{D}^1 \cap C^1(\mathbb{R})$: f'–abs. continuous in $\mathbb{R}$, $f''-$ bounded a.e. in $\mathbb{R}$,

I. Karatzas, S. E. Shreve, 1991, page 219, Problem 7.3

$f(X(t))$–standard Itô formula holds for $f \in \mathcal{D}^2$, X–continuous semimartingale.

Classes of input data $(\sigma_1, \sigma_2, a, b, \eta)$*:*

- *Regular case:*

(F1) $F_{\mathrm{reg}}^{\mathrm{mul},\varrho} = \mathcal{F}_{\mathrm{reg}}^{\varrho} \times \mathcal{F}_{\mathrm{reg}}^{\varrho} \times \mathcal{D}^1 \times \mathcal{D}^1 \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

(F2) $F_{\mathrm{reg}}^{\mathrm{add},\varrho} = \mathcal{F}_{\mathrm{reg}}^{\varrho} \times \mathcal{F}_{\mathrm{reg}}^{\varrho} \times \mathcal{D}^2 \times \{f \equiv const\} \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

- *Singular case:*

(F3) $F_{\mathrm{sng},p}^{\mathrm{mul},\varrho} = \mathcal{F}_{\mathrm{sng},p}^{\varrho} \times \mathcal{F}_{\mathrm{sng},p}^{\varrho} \times \mathcal{D}^1 \times \mathcal{D}^2 \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

(F4) $F_{\mathrm{sng},p}^{\mathrm{add},\varrho} = \mathcal{F}_{\mathrm{sng},p}^{\varrho} \times \mathcal{F}_{\mathrm{sng},p}^{\varrho} \times \mathcal{D}^2 \times \{f \equiv const\} \times \{\eta \mid \mathbb{E}\eta^2 \leqslant \bar{L}\}$,

$\bar{L} > 0$, $F_{\mathrm{reg}}^{\mathrm{add},\varrho} \subset F_{\mathrm{reg}}^{\mathrm{mul},\varrho} \subset F_{\mathrm{sng},p}^{\mathrm{mul},\varrho}$, $F_{\mathrm{reg}}^{\mathrm{add},\varrho} \subset F_{\mathrm{sng},p}^{\mathrm{add},\varrho}$,

$F_{\mathrm{reg}}^{\mathrm{add},\varrho} \Rightarrow$ weaker assumptions than in Hofmann, Müller–Gronbach, Ritter (2001).

Algorithm:

$$\mathcal{A}(\sigma_1, \sigma_2, a, b, \eta, W) = \psi(N(\sigma_1, \sigma_2, a, b, \eta, W)), \quad \psi : \mathbb{R}^n \to \mathbb{R},$$

Ψ_n^*– adapt (a, b),Ψ_n^{**}–adapt $(a, b, \sigma_1, \sigma_2)$, $\Psi_n^* \subset \Psi_n^{**}$,

Model of error:

$$e(\mathcal{A}, (\sigma_1, \sigma_2, a, b, \eta)) = \left(\mathbb{E}\Big(X(T) - \mathcal{A}\Big(\sigma_1, \sigma_2, a, b, \eta, W\Big)\Big)^2\right)^{1/2}, \tag{2}$$

$$e(\mathcal{A}, \mathcal{G}) = \sup_{(\sigma_1, \sigma_2, a, b, \eta) \in \mathcal{G}} e(\mathcal{A}, (\sigma_1, \sigma_2, a, b, \eta)), \tag{3}$$

$$e_n^{\diamond}(\mathcal{G}) = \inf_{\mathcal{A} \in \Psi_n^{\diamond}} e(\mathcal{A}, \mathcal{G}), \quad \diamond \in \{\star, \star\star\} \tag{4}$$

Known results:

- $\left(\sigma_2 \equiv 0 \vee b \equiv 0\right) \wedge a \equiv const \Rightarrow$

$$X(T) = \int_0^T \sigma_1(t)dt, \tag{5}$$

Plaskota, Wasilkowski (2005),

- $\left(\sigma_2 \equiv 0 \vee b \equiv 0\right) \wedge \sigma_1 \equiv const \Rightarrow$

$$dX(t) = a(X(t))dt, \tag{6}$$

Kacewicz, P. (2008),

Known results:

- $\Big(\sigma_1 \equiv 0 \vee a \equiv 0\Big) \wedge b \equiv const \Rightarrow$

$$X(T) = \int_0^T \sigma_2(t)dW(t), \tag{7}$$

  Woźniakowski, Wasilkowski (2001), P. (2009, 2010),
- Pointwise approximation of (1) with regular coefficients (asymptotic case) – Müller–Gronbach (2004),
- Weak convergence of the Euler scheme for SDEs with discontinuous a and b – Yan (2002).

$n \in \mathbb{N}_+$, $0 = t_0 < t_1 < \ldots < t_n = T$, $\Delta t_i = t_{i+1} - t_i$,
$\Delta t_{\max} = \max\limits_{0 \leqslant i \leqslant n-1} \Delta t_i$,
$\Delta W_i = W(t_{i+1}) - W(t_i)$,

Euler algorithm X^E (Kloeden, Platen, 1992):

$$\begin{cases} \hat{X}^E(0) = \eta, \\ \hat{X}^E(t_{i+1}) = \hat{X}^E(t_i) + \sigma_1(t_i)a(\hat{X}^E(t_i))\Delta t_i + \sigma_2(t_i)b(\hat{X}^E(t_i))\Delta W_i, \end{cases}$$

$i = 0, 1, \ldots, n-1$,

$$X^E(\sigma_1, \sigma_2, a, b, \eta, W) = \hat{X}^E(T). \tag{8}$$

$X^E \in \Psi^*_{c_1 n}$, $c_1 > 0$

Theorem 1. (Error of the Euler algorithm X^E)

If $\Delta t_{\max} = T/n$ *then:*

$$e(X^E, F^{\mathrm{add},\varrho}_{\mathrm{sng},p}) \leqslant C_1\Big(1+\sum_{j=1}^{p}|\Delta^j_{\sigma_1}|\Big)n^{-\varrho} + C_2\Big(\sum_{j=1}^{p}|\Delta^j_{\sigma_2}|\Big)n^{-1/2}, \quad (9)$$

and

$$e(X^E, F^{\mathrm{mul},\varrho}_{\mathrm{sng},p}) \leqslant K_1\Big(1+\sum_{j=1}^{p}|\Delta^j_{\sigma_1}|\Big)n^{-\varrho} + K_2\Big(1+\sum_{j=1}^{p}|\Delta^j_{\sigma_2}|\Big)n^{-1/2}. \quad (10)$$

Positive constants C_1, C_2 (K_1, K_2) *depend only on the parameters of the class* $F^{\mathrm{add},\varrho}_{\mathrm{sng},p}$ ($F^{\mathrm{mul},\varrho}_{\mathrm{sng},p}$) *and* T*.*

Proof $\Rightarrow$ Itô formula + local time formula for contionuous semimartingales.

Results:

- $\varrho \in (0, 1]$

$$e_n^{**}(F_{\mathrm{reg}}^{\mathrm{add},\varrho}) = \Theta(n^{-\varrho}), \tag{11}$$

$$C_1 n^{-\varrho} \leqslant e_n^{**}(F_{\mathrm{reg}}^{\mathrm{mul},\varrho}) \leqslant C_2 n^{-\min\{1/2,\varrho\}}, \tag{12}$$

$$C_1 n^{-\varrho} \leqslant e_n^{**}(F_{\mathrm{sng},1}^{\mathrm{mul},\varrho}) \leqslant C_2 n^{-\min\{1/2,\varrho\}}, \tag{13}$$

- $\varrho \in (0, 1], p \geqslant 1$

$$e_n^{*}(F_{\mathrm{sng},p}^{\mathrm{add},\varrho}) = \Theta(n^{-\min\{1/2,\varrho\}}), \tag{14}$$

$$e_n^{*}(F_{\mathrm{sng},p}^{\mathrm{mul},\varrho}) = \Theta(n^{-\min\{1/2,\varrho\}}), \tag{15}$$

- $\varrho \in (0, 1/2], p \geqslant 1$

$$e_n^{**}(F_{\mathrm{sng},p}^{\mathrm{add},\varrho}) = \Theta(n^{-\varrho}), \tag{16}$$

$$e_n^{**}(F_{\mathrm{sng},p}^{\mathrm{mul},\varrho}) = \Theta(n^{-\varrho}), \tag{17}$$

- $\varrho \in (1/2, 1], p \geqslant 2$

$$e_n^{**}(F_{\mathrm{sng},p}^{\mathrm{add},\varrho}) = \Theta(n^{-1/2}), \tag{18}$$

$$e_n^{**}(F_{\mathrm{sng},p}^{\mathrm{mul},\varrho}) = \Theta(n^{-1/2}). \tag{19}$$

- $\varrho \in (1/2, 1],\ p(\sigma_2) = 1,$

$$\tilde{F}_{\mathrm{sng},p,1}^{\mathrm{add},\varrho} = \Big\{ \Big(\sigma_1, \sigma_2, a, b, \eta \Big) \in F_{\mathrm{sng},p}^{\mathrm{add},\varrho} \mid \sigma_2 \in \mathcal{F}_{\mathrm{sng},1}^{\varrho} \Big\}, \tag{20}$$

$$e_n^{**}(\tilde{F}_{\mathrm{sng},p,1}^{\mathrm{add},\varrho}) = \Theta(n^{-\varrho}), \tag{21}$$

X^{E*} :

(i) detection of the unknown singularity of σ_2,
(ii) suitable modification of the starting grid,
(iii) X^E on new mesh.

- $\varrho \in (1/2, 1], p(\sigma_2) \geqslant 2,\ \delta > 0,$

$$\tilde{F}^{\mathrm{add},\varrho,\delta}_{\mathrm{sng},p} = \Big\{ \Big(\sigma_1, \sigma_2, a, b, \eta\Big) \in F^{\mathrm{add},\varrho}_{\mathrm{sng},p} \mid \sigma_2 \in \mathcal{F}^{\varrho,\delta}_{\mathrm{sng},p} \Big\},$$
$$\mathcal{F}^{\varrho,\delta}_{\mathrm{sng},p} = \{ g \in \mathcal{F}^{\varrho}_{\mathrm{sng},p} \mid \min_{0 \leqslant q \leqslant p-1} |s_g^{q+1} - s_g^q| > \delta \},$$

$\Delta t_{\max} = T/n,\ n_0 = \lceil T/\delta \rceil, n \geqslant n_0 \Rightarrow$ separated singularities,

$$e^{**}_{M(n)}(\tilde{F}^{\mathrm{add},\varrho,\delta}_{\mathrm{sng},p}) = O(n^{-\varrho}), \tag{22}$$

cost: $C_1 n \leqslant M(n) \leqslant C_2 n \log_2 n,$

Hp.

For $\varrho \in (1/2, 1]$

$$e_n^{**}(F_{\mathrm{reg}}^{\mathrm{mul},\varrho}) = \Theta(n^{-1/2}), \tag{23}$$

$$e_n^{**}(F_{\mathrm{sng},1}^{\mathrm{mul},\varrho}) = \Theta(n^{-1/2}), \tag{24}$$

and Euler algorithm X^E–optimal.

Asymptotic setting and SDEs with additive noise ($b \equiv const$):

$$g \in \mathcal{F}^{\varrho}_{\mathrm{sng},p},\ g_{\epsilon,u} = g + \epsilon \mathbf{1}_{[0,u)},\ u \in [0,T),\ g_{\epsilon,T} = g + \epsilon,$$
$$X_{\epsilon,u}(T) = X(\sigma_1, \sigma_{2,\epsilon,u}, a, T),$$
$$\mathcal{A}_{n,\epsilon,u}(\sigma_1, \sigma_2, a) = \mathcal{A}_n(\sigma_1, \sigma_{2,\epsilon,u}, a, \eta, W)$$

Proposition 1.

Let us consider sequence of algorithms $\{\mathcal{A}_n\}_{n=1}^{\infty}$, in which every $\mathcal{A}_n$ uses nonadaptive information about σ_2 with total cardinality n. Then for all $a \in \mathcal{D}^1$, $\sigma_1, \sigma_2 \in \mathcal{F}^{\varrho}_{\mathrm{sng},p}$ and for all $\epsilon \neq 0$ the set

$$A = \Big\{ u \in [0,T] \mid \lim_{n\to\infty} n^{1/2} \cdot \|X_{\epsilon,u}(T) - \mathcal{A}_{n,\epsilon,u}(\sigma_1,\sigma_2,a)\|_{L^2(\Omega)} = 0 \Big\},$$

is of Lebesgue measure zero.

Idea of proof $\Rightarrow$ Wasilkowski, Plaskota (2005).

What next?

- General Itô–Taylor schemes for SDEs with irregular σ_1 and σ_2,
- Lévy or fBm noise.

Bibliography

- P. HERTLING, *Nonlinear Lebesgue and Itô Integration Problems of High Complexity*, J.Complexity 17 (2001), 366-387.
- P. E. KLOEDEN, E. PLATEN, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag Berlin, Heidelber 1992
- I. KARATZAS, S. E. SHREVE, *Brownian Motion and Stochastic Calculus*, 2nd edition, Springer - Verlag, New York, 1991.
- B. KACEWICZ, P. PRZYBYŁOWICZ, *Optimal adaptive solution of initial-value problems with unknown singularities*, Journal of Complexity (2008), vol.24, Issue 4 - s. 455–476,

Bibliography

- T. Müller–Gronbach, *Optimal pointwise approximation of SDEs based on Brownian motion at discrete points*, Ann. Appl. Probab. 14(4), 1605–1642 (2004),
- L. Plaskota, G.W.Wasilkowski, *Adaption allows efficient integration of functions with unknown singularities*, Numer. Math. 102 (2005) 123–144.
- L. Plaskota, G. W. Wasilkowski, Y. Zhao, *The power of adaption for approximating functions with singularities*, Math. Comp. 77 (2008), 2309-2338.
- L. Plaskota, G. W. Wasilkowski, *Uniform approximation of piecewise r–smooth and globally continuous funcitons*, SIAM J. Numer. Analysis 47 (2009), 762–785,

Bibliography

- P. Przybyłowicz, *Linear information for the approximation of the Itô integrals*, Numer. Algorithms 52 (2009), 677-699,
- P. Przybyłowicz, *Adaptive Itô–Taylor algorithm can optimally approximate the Itô integrals of singular functions*, J. Comp. and Appl. Math, doi:10.1016/j.cam.2010.05.033, 2010,
- K. Ritter,*Average-Case Analysis of Numerical Problems*, Springer-Verlag Berlin Heidelberg 2000,
- G.W.Wasilkowski, *Information of varying cardinality*, J.Complexity 2, 204-228, 1986,
- G. W. Wasilkowski, H.Woźniakowski, *On the Complexity of Stochastic Interation*,Math. Comp. 70 (2001), 685-698.