Nonlinear Approximation and Besov Regularity of a Class of Random Fields

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Outline

I. The Stochastic Model

II. Linear and Best N-Term Approximation

III. Besov Regularity

Joint work with

P. Cioica, S. Dahlke, S. Kinzel (all from Marburg),

F. Lindner, R. Schilling (both from Dresden),

and T. Raasch (Mainz).

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I. The Stochastic Model

Given

- a bounded Lipschitz domain $D \subset \mathbb{R}^d$,
- a Riesz basis $(\psi_{j,k})_{j \in \mathbb{N}_0,\, k \in \nabla_j}$ for $L_2(D)$ with $\#\nabla_j \asymp 2^{jd}$,

  i.e., for

$$f = \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} c_{j,k} \cdot \psi_{j,k} \in L_2(D)$$

  it holds

$$\|f\|^2 \asymp \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} c_{j,k}^2$$

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Let $Y_{j,k}$, $Z_{j,k}$ be independent with

$$Y_{j,k} \sim B(1, 2^{-\beta jd}), \qquad Z_{j,k} \sim N(0, 2^{-\alpha jd}).$$

Define

$$X = \sum_{j=0}^{\infty} \sum_{k\in\nabla_j} Y_{j,k} Z_{j,k} \cdot \psi_{j,k}.$$

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Note that X is Gaussian iff $\beta = 0$.

II. Linear and Best N-Term Approximation

Put

$$e(\widehat{X}) = \left(E\|X - \widehat{X}\|_{L_2(D)}^2\right)^{1/2}.$$

Linear approximation errors

$$e_N^{\mathrm{lin}} = \inf e(\widehat{X}_N)$$

with infimum over all $\widehat{X}_N$ such that

$$\dim \operatorname{span}(\widehat{X}_N(\Omega)) \leq N,$$

see *Mathé (1990)*.

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Theorem 1 *C-D-K-L-R-R-S (2010)*

$$e_N^{\text{lin}} \asymp N^{-\frac{\alpha+\beta-1}{2}}.$$

For $f = \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} c_{j,k} \cdot \psi_{j,k} \in L_2(D)$

$$\eta(f) = \#\{(j,k) : c_{j,k} \neq 0\}.$$

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$$e_N = \inf e(\widehat{X}_N)$$

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and

$$e_N^{\text{avg}} = \inf e(\widehat{X}_N)$$

with infimum over all $\widehat{X}_N$ such that

$$E(\eta(\widehat{X}_N)) \leq N.$$

Theorem 2 *C-D-K-L-R-R-S (2010)*

$$e_N^{\text{avg}} \preceq \begin{cases} N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}, & \text{if } \beta < 1 \\ 2^{-\alpha d N/2}, & \text{if } \beta = 1, \end{cases}$$

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and for every $\varepsilon > 0$

$$e_N \preceq \begin{cases} N^{-\frac{\alpha+\beta-1}{2(1-\beta)} + \varepsilon}, & \text{if } \beta < 1 \\ N^{-1/\varepsilon}, & \text{if } \beta = 1. \end{cases}$$

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Remark

- Simulation of $\widehat{X}_N$ that yields upper bound for e_N^{avg} at cost $N(\ln N)^2$.
- For $\beta < 1$

$$e_{N^{1-\beta}}^{\mathrm{avg}} \preceq e_N^{\mathrm{lin}}.$$

Nonlinear approximation of stochastic processes, $D = [0, 1]$,

- wavelet methods for piecewise stationary processes, *Cohen, d'Ales (1997)*, *Cohen, Daubechies, Guleryuz, Orchard (2002)*,
- free Knot splines for Brownian motion, SDEs, *Kon, Plaskota (2005)*, *Creutzig, Müller-Gronbach, R (2007)*, *Slassi (2010)*,
- free Knot splines for Lévy driven SDEs, *Dereich (2010), Dereich, Heidenreich (2010)*.

III. Besov Regularity

In the sequel $p, q, s > 0$.

Assumption For all $f = \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} c_{j,k} \cdot \psi_{k,k} \in L_2(D)$

$$\|f\|^q_{B^s_q(L_p(D))} \asymp \sum_{j=0}^{\infty} 2^{j(s+d(1/2-1/p))q} \left(\sum_{k \in \nabla_j} c^p_{j,k} \right)^{q/p}$$

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Theorem 3 *C-D-K-L-R-R-S (2010)*

$X \in B^s_q(L_p(D))$ P-a.s. iff

$$\frac{\alpha - 1}{2} + \frac{\beta}{p} > \frac{s}{d},$$

in which case $E\left(\|X\|^q_{B^s_q(L_p(D))}\right) < \infty$.

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Remark See *Abramovich, Sapatinas, Silverman (1998)* and *Bochkina (2006)* for $d = 1$ and $p, q \geq 1$.

Application: Bayesian non-parametric regression.

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Proof: A.s convergence or weak convergence of

$$\sum_{k \in \nabla_j} Y_{j,k} \left(2^{\alpha j d/2} |Z_{j,k}|\right)^p .$$

Work in progress

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Parabolic SPDE

$$dU(t) = \big(\Delta U(t) + A(t, U(t))\big)\, dt + B(t, (U(t))\, dX(t).$$