On the optimal approximation of certain stochastic integrals

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Discrete approximation of stochastic integrals:

- in practice (f. ex. in stochastic finance) one often needs to approximate discretely stochastic integrals of form

$$f(X_1) = \mathbb{E}f(X_1) + \int_{(0,1]} \varphi(t, X_t) dX_t \quad a.s.$$

- we assume deterministic time nets (discretization points)
- error: the difference between the original integral and the discrete approximation (in finance, the difference between the continuously adjusted hedging port folio and the discretely adjusted one)
- optimization problem: minimize the quadratic error if one has n discretization points
- non-linear approximation problem!: optimal time nets depend on f, which is in finance the pay-off function of a European type option.

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Underlying process

Consider the process $X = (X_t)_{t\in[0,1]}$ such that

$$dX_t = \sigma(X_t)dW_t \quad \text{with } X_0 \equiv x_0 \in \mathbb{R}. \tag{1}$$

Process X is obtained through $Y = (Y_t)_{t\in[0,1]}$ given by

$$dY_t = \hat{\sigma}(Y_t)dW_t + \hat{b}(Y_t)dt \quad \text{with } Y_0 \equiv y_0 \in \mathbb{R},$$

with $0 < \epsilon_0 \leq \hat{\sigma} \in \mathcal{C}_b^\infty(\mathbb{R})$ and $\hat{b} \in \mathcal{C}_b^\infty(\mathbb{R})$, in the following two ways:

(a) $y_0 = x_0 \in \mathbb{R}$, $\hat{\sigma} := \sigma$, $\hat{b} := 0$, $X_t := Y_t$,

(b) $y_0 = \log x_0$ with $x_0 > 0$,

$$\hat{\sigma}(y) := \frac{\sigma(e^y)}{e^y}, \quad \hat{b}(y) := -\frac{1}{2}\hat{\sigma}(y)^2, \quad \textit{and } X_t = e^{Y_t}.$$

That means we consider the martingale case (X is a martingale) and have a "good" transition density for X.

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Assumptions on function and time net

Assume that $\mathcal{C}_e$ is the linear space of Borel measurable functions $g : \mathbb{R} \to \mathbb{R}$ such that for some $m > 0$,

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} e^{-m|x|}|g(x)| < \infty$$

We consider functions f such that $f(X_1) = g(Y_1)$ for some $g \in \mathcal{C}_e$.
Actually we are interested in the value at time one and hence we work with

$$\mathcal{C} := \{Z := g(Y_1) : \Omega \to \mathbb{R} \mid g \in \mathcal{C}_e \text{ and } Y \text{ as above}\}.$$

We consider the set of deterministic time nets

$$\mathcal{T}_n := \{\tau = (t_i)_{i=0}^n : \; 0 = t_0 < t_1 < \cdots < t_n = 1\}$$

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Stochastic integral

Suppose $Z = f(X_1)$ and let us define

$$F(t,x) := \mathbb{E}(Z|X_t = x) \quad \text{for all } t \in [0,1).$$

It is known that $F(t,x)$ satisfies the heat equation

$$\frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} = 0 \quad \text{on } [0,1) \times I,$$

where $I = \mathbb{R}$ in case (a) and $I = (0,\infty)$ in case (b). Hence by Itô's formula

$$Z = \mathbb{E}Z + \int_0^1 \frac{\partial F}{\partial x}(t, X_t)dX_t.$$

We are looking for the optimal discrete approximation of this integral.

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Approximation error

Let

$$C(Z,\tau^n,X,v) := Z - \mathbb{E}Z - \sum_{i=1}^{n} v_{i-1}(X_{t_i} - X_{t_{i-1}}), \qquad (2)$$

where

(1) $\mathbb{E}Z + \sum_{i=1}^{n} v_{i-1}(X_{t_i} - X_{t_{i-1}})$ is our approximation of Z

(2) $\tau^n \in \mathcal{T}_n$ is a time net consisting of discretization points,

(3) v_{i-1} is an $\mathcal{F}_{t_{i-1}}$-measurable step function for all $i = 1, ..., n$. ($(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the filtration generated by W.)

We are interested in the L_2-error:

$$a_X(Z,\tau^n) := \inf_{(v_i)_{i=0}^{n-1}} \left(\mathbb{E}|C(Z,\tau^n,X,v)|^2\right)^{\frac{1}{2}}. \qquad (3)$$

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H-functional

We define $H_X : \mathcal{C} \times [0,1) \to [0,\infty)$ by setting

$$H_X Z(t) := \left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} \quad \text{for all } t \in [0,1).$$

The function $H_X Z$ is continuous and increasing with respect to t and it is an essential tool in investigations of the optimal approximation rate.

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Theorem 2.1 (C. and S. Geiss (2004), S. Geiss and Hujo (2006))

Assume that $Z \in \mathcal{C}$ and X is as above.

(1) *Then*

$$\inf_{n\in\mathbb{N}} \inf_{\tau^n \in \mathcal{T}_n} \sqrt{n} a_X(Z, \tau^n) > 0$$

unless Z can be written as $Z = c_0 + c_1 X_1$ a.s., where $c_0, c_1 \in \mathbb{R}$.

(2) *Let $\tau_n^\theta = (1 - (1 - \frac{i}{n})^{\frac{1}{\theta}})_{i=0}^n$, where $\theta \in (0,1]$. If X is the BM or gBM,*

$$a_X(Z, \tau_n^\theta) \le \frac{c}{\sqrt{n}} \quad \text{for all } n \in \mathbb{N},$$

if and only if

$$\int_0^1 (1-t)^{1-\theta} H_X Z(t)^2 dt < \infty.$$

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Questions

Is it possible to characterize the optimal approximation rate

$$a_n^X(Z) := \inf_{\tau_n \in \mathcal{T}_n} a_X(Z, \tau^n) := \inf_{\tau_n \in \mathcal{T}_n} \inf_{(v_i)_{i=0}^{n-1}} \left(\mathbb{E}|C(Z, \tau^n, X, v)|^2\right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{n}} \,?$$

Can one give some conditions under which the approximation rate is

$$0 < \sup_{n \in \mathbb{N}} \sqrt{n^\beta} a_n^X(Z) < \infty,$$

where $\beta \in (0, 1)$?

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Characterization of $\inf a_X(Z,\tau) \leq \frac{c}{\sqrt{n}}$

Theorem 3.1

Let $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if X is BM or geometric BM) and X be as in (1). Then

$$\sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) < \infty$$

if and only if

$$\int_0^1 H_X Z(t) dt < \infty.$$

How to set the time nets?

If the assumptions of Theorem 3.1 are satisfied and $A := \int_0^1 H_X Z(t)dt < \infty$, then there exists a constant $c_\sigma > 0$ such that

$$a_n^X(Z) \leq \frac{c_\sigma A}{\sqrt{n}} \quad \textit{for all } n \in \mathbb{N}.$$

This rate can be obtained by regular sequences generated by $H_X Z(t)$, i.e. the time nets $\tau^n = (t_i^n)_{i=0}^n$ for which

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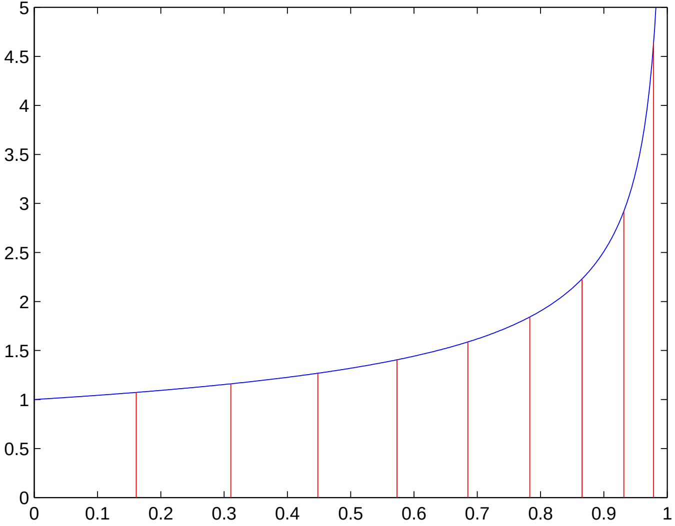
Optimal time nets



From the point of view of approximation theory:

Definition 3.2

We say that a random variable $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if X is BM or gBM) belongs to $\mathcal{A}$ if and only if

$$\|Z\|_{\mathcal{A}} := \|Z\|_{L_2} + \sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) < \infty$$

It is possible that $a_n^X(Z_1)$ and $a_n^X(Z_2)$ are obtained using different time nets and it seems that $\mathcal{A}$ is not a normed space.

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Definition 3.3

The random variable $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if X is BM or gBM) belongs to $\mathcal{H}$ if and only if

$$\|Z\|_{\mathcal{H}} := \|Z\|_{L_2} + \int_0^1 H_X Z(t) dt < \infty.$$

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Theorem 3.1 implies that $\mathcal{A} = \mathcal{H}$ and we can show that

$$\|Z\|_{\mathcal{A}} \sim_c \|Z\|_{\mathcal{H}}$$

for some $c \geq 1$ depending on σ only. This is interesting, since $\|Z\|_{\mathcal{H}}$ is a norm, but $\|Z\|_{\mathcal{A}}$ does not seem to be a norm.

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What about optimal rate $\frac{1}{\sqrt{n^\beta}}$?

- It was shown by Hujo (2006) that for any sequence $\eta(n)$, with $\lim_{n\to\infty} \eta(n) = \infty$, there exists a random variable $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and a constant $c \geq 1$ such that

$$a_n^W(Z) \geq \frac{1}{c\eta(n)} \quad \text{for all } n \geq 2.$$

- For all known explicit functions the optimal rate is $\frac{1}{\sqrt{n}}$.
- Could it be possible to find conditions for the approximation rate $\frac{1}{\sqrt{n^\beta}}$?

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- Could it be possible to find conditions for the approximation rate $\frac{1}{\sqrt{n^\beta}}$?

What about optimal rate $\frac{1}{\sqrt{n^\beta}}$?

- It was shown by Hujo (2006) that for any sequence $\eta(n)$, with $\lim_{n\to\infty} \eta(n) = \infty$, there exists a random variable $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and a constant $c \geq 1$ such that

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Conditions for approximation rate $\frac{c}{\sqrt{n^\beta}}$

Theorem 3.4

Let $Z \in \mathcal{C}$, X be as in (1) (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if X is BM or gBM), and $\alpha \in (\frac{1}{2}, 1)$. Then

(1) *If there exists a constant $c_1 \geq 1$ such that*

$$H_X Z(t) \leq c_1 \frac{(1 - \log(1-t))^{-\alpha}}{1-t} \quad \textit{for all } t \in [0,1),$$

then

$$a_n^X(Z) \leq \frac{c}{\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N}$$

where $c = c(\alpha, \sigma, c_1) \geq 1$.

(2) If there exists $s \in [0,1)$ and a constant $c_2 \geq 1$ such that

$$H_X Z(t) \geq \frac{(1-\log(1-t))^{-\alpha}}{c_2(1-t)} \quad \text{for all } t \in [s,1),$$

then

$$a_n^X(Z) \geq \frac{1}{c\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N},$$

where $c = c(\alpha, \sigma, c_2) \geq 1$.

About the proofs

In the end the problem is deterministic due to the following result:

Theorem 3.5 (S. Geiss (2002))

Let X be a stochastic process as in (1), $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if $X \in \{W, S\}$) and $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$. Then

$$a_X(Z, \tau) \sim_c \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H_X Z(t)^2 dt \right)^{\frac{1}{2}}$$

where $c \geq 1$ is an absolute constant depending on σ only.

This means that it is sufficient to investigate the quantity

$$A_n(H) := \inf_{\tau \in \mathcal{T}_n} \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H(t)^2 dt \right)^{\frac{1}{2}},$$

where $H : [0, 1) \to [0, \infty)$ is increasing.

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References

- Geiss, C., Geiss, S. (2004):*On approximation of a class of stochastic integrals and interpolation*, Stochastics and Stochastics Reports 76 (2004) 339-362.
- Geiss, S., (2002): *Quantitative approximation of certain stochastic integrals* Stochastics and Stochastics Reports 73 (2002) 339-362.
- Geiss, S., Hujo, M. (2007): *Interpolation and approximation in $L_2(\gamma)$*, Journal of Approximation Theory 144 (2007) 213-232.
- Hujo, M., (2006): *Is the approximation rate for certain stochastic integrals always $1/\sqrt{n}$?*, Journal of Theoretical Probability 19 (2006) 190-203.
- Seppälä, H.: *On the optimal approximation rate of certain stochastic integrals*, Journal of Approximation Theory (2010), doi:10.1016/j.jat2010.04.008.