Approximation with general information versus function evaluations

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Overview

1. The Problem
2. What do we know?
3. Results
4. Overall idea for the proof

The Problem

We want to approximate the embedding operator

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The optimal algorithms $A_n : H \to L_p(X)$ that use n linear functionals or n function evaluations are linear. That means

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Measuring the error

Definition (Approximation numbers and sampling numbers)

For $A : H \to L_p(X)$, $A(f) := f$ we define the approximation numbers and sampling numbers as

$$a_n(H \subset L_p(X)) := \inf_{\substack{\alpha_1,\dots,\alpha_n \in H' \\ h_1,\dots,h_n \in L_p}} \sup_{\substack{f \in H \\ \|f\|_H \leq 1}} \left\| f - \sum_{i=1}^{n} \alpha_i(f) h_i \right\|_p ,$$

$$g_n(H \subset L_p(X)) := \inf_{\substack{x_1,\dots,x_n \in X \\ h_1,\dots,h_n \in L_p}} \sup_{\substack{f \in H \\ \|f\|_H \leq 1}} \left\| f - \sum_{i=1}^{n} f(x_i) h_i \right\|_p .$$

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Complexity of an approximation problem

Definition (Rate of convergence)

The rate of convergence of a null sequence (c_n) is defined as

$$r(c_n) := \sup\{\beta \in \mathbb{R} : \lim_{n\to\infty} c_n n^\beta = 0\}$$

Example

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The BIG question

Is $r(a_n) = r(g_n)$?

What do we know?

Theorem (Positive results)

For $p = 2$ and $r(a_n) > \frac{1}{2}$ we have

$$r(g_n) \geq \frac{2r(a_n)}{2r(a_n) + 1} r(a_n) > \frac{1}{2} r(a_n) .$$

(Kuo, Wasilkowski, Woźniakowski, 2008)

Furthermore: For all known examples where $p = 2$ and $r(a_n) > \frac{1}{2}$ we have

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What do we know?

Theorem (Negative results)

There is a Hilbert space embedding $H \subset \ell_2$ with

$$r(a_n) = \frac{1}{2} \qquad \text{and} \qquad r(g_n) = 0\,.$$

(Hinrichs, Novak, Víbiral, 2008)

Results

Theorem (Main result)

For $p \in [1, \infty)$ there exists an embedding $H \subset \ell_p$ with

$$r(a_n) = \min\left\{\frac{1}{2}, \frac{1}{p}\right\} \qquad \text{and} \qquad r(g_n) = 0\,.$$

Overall idea for the proof (step 1)

Get sufficiently bad sampling numbers for finite dimsional examples: For $N \in \mathbb{N}$ let $H_{N,\delta,\varepsilon} := \mathbb{R}^N$ with

$$\|x\|^2_{H_{N,\delta,\varepsilon}} := \frac{1}{\delta^2}(x,y)^2 + \frac{1}{\varepsilon^2}\|(x-(x,y)y\|_2^2 \,,$$

where $y = N^{-1/2}(1,\ldots,1) \in \mathbb{R}^N$. For instance for $p=2$ this yields

$$a_n(H_{N,\delta,\varepsilon} \subset \ell_2^N) = \begin{cases} \delta & \text{for } n=0, \\ \varepsilon & \text{for } n>0, \end{cases}$$

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Lemma

Let $p \geq 2$. Furthermore, let $(\kappa_M)_{M\in\mathbb{N}}$ and $(\lambda_M)_{M\in\mathbb{N}}$ be convergent series of real numbers with $\kappa := \lim_{M\to\infty} \kappa_M > \lim_{M\to\infty} \lambda_M =: \lambda$ If for every $M \in \mathbb{N}^+$ there are an $N \in \mathbb{N}^+$ and an embedding of a Hilbert space $H_M \subset \ell_p^N$, such that

$$a_n(H_M \subset \ell_p^N) \leq \frac{1}{(M+n)^{\kappa_M}} \qquad \text{for } \underline{\text{all}}\ n \in \{0,\dots,N\},$$

$$g_n(H_M \subset \ell_p^N) \geq \frac{1}{n^{\lambda_M}} \qquad \text{for } \underline{\text{some}}\ n \in \{0,\dots,N\},$$

then there exists an embedding of a Hilbert space $H \subset \ell_p$ with

$$r(a_n(H \subset \ell_p)) \geq \kappa > \lambda \geq r(g_n(H \subset \ell_p)).$$

Overall idea for the proof (step 3)

Choose the right parameters N, ε and δ as input for the lemma.

Get the result:

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Open Question.

If $r(a_n) > \min\left\{\frac{1}{2}, \frac{1}{p}\right\}$, does $r(a_n) = r(g_n)$ follow?

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