Jump-adapted discretization schemes for Lévy-driven SDEs

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We are interested in numerical evaluation of

\[ E[f(X_1)], \quad \text{where} \quad X_t = X_0 + \int_0^t h(X_{s-})dZ_s, \quad X \in \mathbb{R}^n \]

where \( Z \in \mathbb{R}^d \) is a pure-jump Lévy process:

\[ Z_t = \gamma t + \int_0^t \int_{|y| \leq 1} y \hat{N}(dy, ds) + \int_0^t \int_{|y| > 1} y N(dy, ds) \]
The Euler scheme with constant time step

\[
\hat{X}_{i+1}^n = \hat{X}_i^n + h(\hat{X}_i^n)\left(Z_{i+1}^n - Z_i^n\right)
\]

has the convergence rate (Protter and Talay ’97)

\[
|E[f(X_1)] - E[f(\hat{X}_1^n)]| \leq \frac{C}{n}
\]

but suffers from two difficulties

- The increments of \( Z \) cannot usually be simulated in closed form;
- A large jump in \( Z \) between two discretization dates may lead to a large discretization error.
Jump-adapted discretization

A natural idea to solve both problems, due to Rubenthaler ’03, is

- Approximate $Z$ with a compound Poisson process

$$Z^{\epsilon}_t := \gamma^{\epsilon} t + \int_0^t \int_{|y| > \epsilon} y N(dy, ds), \quad \gamma^{\epsilon} = \gamma - \int_{\epsilon < |y| \leq 1} y \nu(dy).$$

- Apply the Euler scheme at every jump time of $Z^{\epsilon}$.

The convergence rate may be computed in terms of expected number of discretization dates, proportional to $\lambda^{\epsilon} = \int_{|y| \geq \epsilon} \nu(dy)$.

This rate may range from very good to very bad (for $Z$ of infinite variation) because

- The variance of small jumps may go to zero very slowly;
- The drift $\gamma^{\epsilon}$ may explode as $\epsilon \to 0$. 

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Taking into account the structure of \( Z \)

- View the Lévy process between the jumps of \( Z^\varepsilon \) as a deterministic ODE perturbed by noise (small jumps).
- Approximate small jumps by Brownian motion (Asmussen and Rosinski ’01):

\[
\left( \int_0^t \int_{|z| \leq \varepsilon} y \tilde{N}(dy, ds) \right)_{0 \leq t \leq 1} \approx (W^\varepsilon_t)_{0 \leq t \leq 1}
\]

where \( W^\varepsilon \) is a \( d \)-dimensional BM with covariance

\[
\Sigma^\varepsilon_{ij} = \int_{|y| \leq \varepsilon} y_i y_j \nu(dy).
\]

- Solve the deterministic ODE explicitly, or with a higher-order scheme, which is easy to construct.
- Expand the solution to the SDE around the explicit solution of the ODE.
Approximating the Lévy process

- Levy process
- Compound Poisson approximation

- Gaussian correction
- Compound Poisson approximation
Approximating the SDE: ODE solution
Towards jump-adapted schemes

Approximating the SDE: Gaussian correction
Start by replacing the small jumps of $Z$ with a Brownian motion:

$$d\bar{X}_t = h(\bar{X}_{t-})\{\gamma_\varepsilon dt + dW_\varepsilon + dZ_\varepsilon\},$$

where $W_\varepsilon$ is a $d$-dimensional Brownian motion with covariance matrix $\Sigma_\varepsilon$. This process can also be written as

$$\bar{X}(t) = \bar{X}(\eta_t) + \int_{\eta_t}^t h(\bar{X}(s)) \, dW_\varepsilon(s) + \int_{\eta_t}^t h(\bar{X}(s)) \gamma_\varepsilon ds,$$

$$\bar{X}(T_i^\varepsilon) = \bar{X}(T_i^\varepsilon -) + h(\bar{X}(T_i^\varepsilon -)) \Delta Z(T_i^\varepsilon),$$
Consider a family of processes \((Y^\alpha)_{0 \leq \alpha \leq 1}\) defined by

\[
Y^\alpha(t) = \bar{X}(\eta_t) + \alpha \int_{\eta_t}^{t} h(Y^\alpha(s)) \, dW^\varepsilon(s) + \int_{\eta_t}^{t} h(Y^\alpha(s)) \, \gamma^\varepsilon ds
\]

Our idea is to replace the process \(\bar{X} := Y^1\) with its first-order Taylor approximation:

\[
\bar{X}(t) \approx Y^0(t) + \frac{\partial}{\partial \alpha} Y^\alpha(t)|_{\alpha=0}.
\]
The new approximation \( \tilde{X} \) is defined by

\[
\tilde{X}(t) = Y_0(t) + Y_1(t), \quad t > \eta,
\]

\[
\tilde{X}(T_i^\varepsilon) = \tilde{X}(T_i^\varepsilon -) + h(\tilde{X}(T_i^\varepsilon -)) \Delta Z(T_i^\varepsilon),
\]

\[
Y_0(t) = \tilde{X}(\eta_t) + \int_{\eta_t}^{t} h(Y_0(s)) \gamma_\varepsilon ds
\]

\[
Y_1(t) = \int_{\eta_t}^{t} \frac{\partial h}{\partial x_i}(Y_0(s)) Y_1^i(s) \gamma_\varepsilon ds + \int_{\eta_t}^{t} h(Y_0(s)) dW^\varepsilon(s)
\]

where we used the Einstein convention for summation over repeated indices.
Computing $Y_0$ and $Y_1$

- $Y_0$ is the solution of an ODE and can be computed, e.g., by a 4th order Runge-Kutta scheme.
- Conditionally on $(T^ε_i)_{i≥1}$, the random vector $Y_1(t)$ is Gaussian with mean zero, and we only need its terminal covariance.
- Its covariance matrix $Ω(t)$ satisfies the linear equation

$$Ω(t) = \int_{η_t}^{t} (Ω(s)M(s) + M^⊥(s)Ω^⊥(s) + N(s))ds$$

where $M^⊥$ denotes the transpose of the matrix $M$ and

$$M_{ij}(t) = \frac{∂h_{ik}(Y_0(t))}{∂x_j}γ^k_ε \quad \text{and} \quad N(t) = h(Y_0(t))Σ^εh^⊥(Y_0(t)).$$

- In one dimension, the solution is $Ω(t) = Σ^ε h^2(Y^0_t)(t - η_t)$. 

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The convergence rate

\((H_n)\) \(f \in C^n, h \in C^n f^{(k)} \) and \(h^{(k)} \) are bounded for \(1 \leq k \leq n \) and 
\[ \int z^{2n} \nu(dz) < \infty. \]

\((H'_n)\) \(f \in C^n, h \in C^n, h^{(k)} \) are bounded for \(1 \leq k \leq n \), \(f^{(k)} \) have at most polynomial growth for \(1 \leq k \leq n \) and 
\[ \int |y|^k \nu(dz) < \infty \] for all \(k \geq 1\).

- Assume \((H_3)\) or \((H'_3)\). Then

\[ |E[f(\hat{X}_1) - f(X_1)]| \leq C \left( \frac{\| \Sigma \|}{\lambda_{\varepsilon}} (\| \Sigma \| + |\gamma_{\varepsilon}|) + \int_{|y| \leq \varepsilon} |y|^3 \nu(dy) \right) \]

- Assume \((H_4)\) or \((H'_4)\) and \(\nu(dy) = (1 + \xi(y))\nu_0(dy)\), where \(\nu_0\) is a symmetric measure and \(\xi(y) = O(y)\). Then

\[ |E[f(\hat{X}_1) - f(X_1)]| \leq C \left( \frac{\| \Sigma \|}{\lambda_{\varepsilon}} + \int_{|y| \leq \varepsilon} |y|^4 \nu(dy) \right). \]
Worst-case bounds

For general Lévy measures,

\[ |E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-\frac{1}{2}}), \]

and in the locally symmetric case,

\[ |E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-1}). \]

- For all known examples, the convergence rates are better.
Stable-like behavior and other examples

- Assume that $\nu$ has a density $\nu(z) = \frac{g(z)}{|z|^{1+\alpha}}$ with $g$ bounded near zero. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_{\varepsilon}^{1 \frac{3}{\alpha}} \vee (-\frac{2}{\alpha})),$$

and if the Lévy measure is symmetric near zero (CGMY),

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_{\varepsilon}^{-\frac{2}{\alpha}}).$$

- The NIG process has a symmetric stable-like Lévy measure with $\alpha = 1$

$$\Rightarrow |E[f(\hat{X}_1) - f(X_1)]| \leq O(1/\lambda_{\varepsilon}^2)$$

- In the VG model, the convergence is exponential:

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C \frac{e^{-2\lambda_{\varepsilon}}}{\lambda_{\varepsilon}}.$$. 
The Libor market model (general case of BGM model) describes joint arbitrage-free dynamics of a set of forward interest rates.

Libor market models with jumps were considered by Jamshidian '99, Glasserman and Kou '03, Eberlein and Özkan ’05, Papapantoleon and Skovmand ’10 and others.

Let $T_i = T_1 + (i - 1)\delta, \ i = 1, \ldots, n + 1$ be a set of dates called tenor dates. The Libor rate $L_i^t$ is the forward rate defined at $t$ for the period $[T_i, T_{i+1}]:$

$$L_i^t = \frac{1}{\delta} \left( \frac{B_t(T_i)}{B_t(T_{i+1})} - 1 \right),$$

where $B_t(T)$ is the price at $t$ of a zero-coupon bond with maturity $T$. 
Introducing jumps

Following Jamshidian ’99, an arbitrage-free dynamics of $n$ forward Libors $L^1_t, \ldots, L^n_t$ can be constructed via the multi-dimensional SDE

$$\frac{dL^i_t}{L^i_t} = \sigma^i(t) dZ_t - \int_{\mathbb{R}^d} \sigma^i(t) z \left[ \prod_{j=i+1}^n \left( 1 + \frac{\delta L^j_t \sigma^j(t) z}{1 + \delta L^j_t} \right) - 1 \right] \nu(dz) dt,$$

where $Z$ is a $d$-dimensional martingale pure jump Lévy process with Lévy measure $\nu$ under the terminal measure $Q$ and $\sigma^i(t)$ are deterministic volatility functions.

Terminal measure: martingale measure for which the last zero-coupon bond $B_t(T_{n+1})$ is the numéraire.
The price of any asset divided by $B_t(T_{n+1})$ is a martingale and in particular the price of an option which pays $H = h(L_{T_1}^1, \ldots, L_{T_1}^n)$ at time $T_1$ (e.g. swaption) is given by

$$
\pi_t(H) = B_t(T_{n+1}) \mathbb{E} \left[ \frac{h(L_{T_1}^1, \ldots, L_{T_1}^n)}{B_{T_1}(T_{n+1})} \bigg| \mathcal{F}_t \right]
$$

$$
= B_t(T_{n+1}) \mathbb{E} \left[ h(L_{T_1}^1, \ldots, L_{T_1}^n) \prod_{i=1}^{n} (1 + \delta L_{T_1}^i) \bigg| \mathcal{F}_t \right].
$$

The price of any such option can therefore be computed by Monte Carlo using the Libor dynamics.
Numerical implementation

- We consider a Libor market model with tenor dates \{5, 6, 7, 8, 9, 10\}, a one-dimensional driving Lévy process and constant volatilities of all Libors (\(\sigma^i(t) \equiv 1\)).
- The initial values are fixed to 15% to emphasize the non-linear effects.
- The differential equations for \(Y_0(t)\) and \(\Omega(t)\) are solved simultaneously by fourth order Runge Kutta.
- The Lévy measure is

\[
C \frac{e^{-\lambda_+ x} 1_{x>0} + e^{-\lambda_- |x|} 1_{x<0}}{|x|^{1+\alpha}}
\]

with \(\lambda_+ = 10\), \(\lambda_- = 20\) and \(\alpha = 0.5\), \(C = 1.5\) (Case 1) or \(\alpha = 1.8\), \(C = 0.01\) (Case 2). Both cases correspond to annualized standard deviation of about 24%.
Sanity check: pricing a zero-coupon

Ratio of estimated to theoretical zero coupon bond price in Case 1 (left) and Case 2 (right). The theoretical convergence rate is $\lambda^{-4}$ (case 1) and $\lambda^{-1.11}$ (case 2). For comparison we also give results of the 0-order scheme (without Brownian approximation, in blue).
Estimated price of an ATM receiver swaption with maturity 5 years in Case 1 (left) and Case 2 (right).
Execution times for the swaption example on a PIII PC without any code optimization.
Conclusion and reference

- New Monte Carlo scheme for weak approximation of multidimensional SDEs driven by pure-jump Lévy processes.
- Does not require sampling from the underlying Lévy process, and often achieves higher rates than the Euler scheme.
- Our theoretical results can potentially be used to improve the rates of multilevel Monte Carlo methods for smooth functionals.