Liberating the Dimension for Weighted Integration and Function Approximation

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Curse of Dimensionality

Many multivariate problems have complexity
exponential in the number d of variables.

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Many multivariate problems have complexity
exponential in the number d of variables.

But

NOT ALL SUCH PROBLEMS!

Now we know that many multivariate problems,
especially defined over weighted spaces, are
TRACTABLE !!!

Since its introduction by H. Woźniakowski in 1994,

Tractability of Multivariate Continuous Problems

has attracted attention of many researches, including

Cools, Creutzing, Dereich, Dick, Doerr, Fasshauer, Gnewuch, Griebel, Heinrich, Hickernell, Hinrichs, Huang, Joe, Kacewicz, Kritzer, Kuo, Kwas, Larcher, L'Ecuyer, Li, Mathe, Müler-Gronbach, Niederreiter, Niu, Novak, Nuyens, Owen, Papageorgiou, Paskov, Pillichshammer, Petras, Plaskota, Przybylowicz, Ritter, Rust, Sinescu, Sloan, Traub, Vybiral, Wang, Wasilkowski, Waterhouse, Werschulz, J. O. Wojtaszczyk, Woźniakowski, Yue, Zhang, Zhao

Three Volume Monograph by Novak and Woźniakowski

Almost all results deal with
arbitrarily large but a **finite** number of variables d

Integration Problem

For $d = 1, 2, \ldots$,

$\mathcal{F}_d$ is a Hilbert space of d-variate functions, $f : [0,1]^d \to \mathbb{R}$

We want to approximate $\mathcal{I}_d(f) = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$

Algorithms use function samples $\mathcal{A}_{d,n}(f) = \sum_{i=1}^{n} f(\boldsymbol{x}_i) \cdot a_i$

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Complexity: $\mathrm{comp}(\varepsilon, d)$ the minimal number of samples needed for error $\leq \varepsilon$.

Hence **cost of computing $f(\boldsymbol{x}_i)$ is independent of $\boldsymbol{x}_i$ and d**

Polynomial Tractability: If there are $C, p, q \geq 0$ such that

$$\mathrm{comp}(\varepsilon, d) \leq C \cdot \varepsilon^{-p} \cdot d^q \qquad \forall \varepsilon > 0, \quad \textbf{for all } d$$

Infinitely Many Variables

There are many computational problems with $d = \infty$ variables, e.g.,

path integrals

Such problems can be approximated by problems with finite d and many existing tractability results could be and are used.

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HOWEVER!

Some results are **IRRELEVANT**

especially negative results, e.g.,

Smolyak's construction, **bad** when d is arbitrary

turns out to be **good** when $d = \infty$

When dealing with $d = \infty$ problems:

Cost of evaluating a d-variate function $f(\boldsymbol{x})$
should depend on d and $\boldsymbol{x}$

Choice of d **should depend on**
the problem and on the error demand ε : $d = d(\varepsilon)$

ACTUALLY: For many problems,
the **number of "active variables" is small:**

$$d(\varepsilon) = o\left(\ln(1/\varepsilon)\right) \quad \text{or even} \quad d(\varepsilon) = \mathcal{O}\left(\sqrt{\ln(1/\varepsilon)}\right)$$

Fixing a sequence $\{d_n\}_n$ with $\lim_n d_n = \infty$
instead of $d(\varepsilon)$ may be **INFERIOR**

LIBERATION OF DIMENSION

that is

LET'S GET FREE FROM d

For Friends using s instead of d

PUT TO RESt

YOUR BELOVED s

LIBERATION SO FAR

Initial attempts for Feynman–Kac–type of integrals in

W. and Woźniakowski [96] and Plaskota, W., and Woźniakowski [00]

Recent approaches for integration:

Creutzing, Dereich, Müller-Gronbach, and Ritter [09],

Gnewuch [10],

Hickernell, Müller-Gronbach, Niu, and Ritter [10],

Hickernell and Wang [01],

Kuo, Sloan, W., and Woźniakowski [09],

Niu and Hickernell [10],

Niu and Hickernell, Müller-Gronbach, and Ritter [10],

However for very special spaces and without sharp bounds

Very recent results to be presented now:

General Spaces and Optimal Results

for function approximation: W. and Woźniakowski [10a] and [10b]

for integration: Plaskota and W. [10]

$d = \infty$ CASE

Functions to be dealt with are of the form:

$$\begin{aligned} f(\boldsymbol{x}) &= f_{\emptyset} + f_{\{1\}}(x_1) + f_{\{2\}}(x_2) + \cdots + f_{\{3,7,9\}}(x_3, x_7, x_9) + \cdots \\ &= \sum_{\mathfrak{u}} f_{\mathfrak{u}}(\boldsymbol{x}), \end{aligned}$$

where $\mathfrak{u} \subset \mathbb{N}$ lists **active variables** in $f_{\mathfrak{u}}$

There are practical problems with
different importance of different groups of variables, e.g., in finance
Caflish, Morokoff and Owen [97], Wang and K.T.Fang [03]

Therefore **weights** $\gamma_{\mathfrak{u}}$ are used
to quantify the importance of variables listed in $\mathfrak{u}$

MORE TECHNICALLY:

QUASI-Reproducing Kernel Hilbert Spaces

Following Kuo, Sloan, W., and Woźniakowski [09]

Let $D \subseteq \mathbb{R}$ and H be a RKHS of functions f:

$$f : D \to \mathbb{R} \quad \text{and} \quad f(x) = \langle f, K(\cdot, x) \rangle_H \quad \text{where } K \text{ is the kernel}$$

ASSUMPTION: $K(a, a) = 0$ for an anchor $a \in D$

EXAMPLE (Wiener kernel):

$K(x, y) = \min(x, y)$ with $a = 0$, where $D = [0, 1]$ or $D = [0, \infty)$

∞-variate domain: $\mathcal{D} = D^{\infty}$, $\boldsymbol{x} = [x_1, x_2, \dots] \in \mathcal{D}$

∞-variate functions:

$$f(\boldsymbol{x}) = \sum_{\mathfrak{u} \subset \mathbb{N}} f_{\mathfrak{u}}(\boldsymbol{x})$$

where $f_{\mathfrak{u}}$ depends on variables in $\mathfrak{u}$, a finite subset of $\mathbb{N}$:

$$f_{\mathfrak{u}} \in H_{\mathfrak{u}} \quad \text{with kernel} \quad K_{\mathfrak{u}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j \in \mathfrak{u}} K(x_j, y_j)$$

SPACE $\mathcal{F}$: completion with respect to weighted inner-product:

$$\langle f, g \rangle_{\mathcal{F}} = \sum_{\mathfrak{u}} \frac{1}{\gamma_{\mathfrak{u}}} \cdot \langle f_{\mathfrak{u}}, g_{\mathfrak{u}} \rangle_{H_{\mathfrak{u}}} \quad \text{for} \quad f = \sum_{\mathfrak{u}} f_{\mathfrak{u}} \qquad g = \sum_{\mathfrak{u}} g_{\mathfrak{u}}$$

Role of weights $\gamma_{\mathfrak{u}}$:

The larger $\gamma_{\mathfrak{u}}$ the more important variables in $\mathfrak{u}$.

E.g., if $\gamma_{\mathfrak{u}} = 0$ then $f_{\mathfrak{u}} = 0$.

Special weights: finite-order weights:

$$\gamma_{\mathfrak{u}} = 0 \quad \text{for all} \quad |\mathfrak{u}| > \omega.$$

Examples:

If $\gamma_{\mathfrak{u}} = 0$ for all $|\mathfrak{u}| > \omega = 1$ then $f(\boldsymbol{x}) = c + \sum_{i=1}^{\infty} f_{\{i\}}(x_i)$

Polynomials of degree 4 have $\gamma_{\mathfrak{u}} = 0$ if $|\mathfrak{u}| > \omega = 4$

Special weights: product weights:

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$$

quantify importance of specific variables $\gamma_j \sim x_j$

Why QUASI-RKHS?

$\mathcal{F}$ is RKHS iff $\sum_{\mathfrak{u}} \gamma_{\mathfrak{u}} \cdot \sup_{x \in D} \left[K(x,x)\right]^{|\mathfrak{u}|} < \infty$

Otherwise,

function sampling $L_{\boldsymbol{x}}(f) = f(\boldsymbol{x})$ is **DISCONTINUOUS!** for some $\boldsymbol{x}$

For Wiener kernel and $D = [0, \infty)$, $\sup_x K(x,x) = \infty$ and $\mathcal{F}$ is **NOT** RKHS. It is only **QUASI**-RKHS

HOWEVER:

Function sampling $L_{\boldsymbol{x}}(f) = f(\boldsymbol{x})$ is always continuous
if $\boldsymbol{x}$ has finitely many active variables

Active variables: x_j is active if $x_j \neq a$

Sampling points used by our algorithms: Given $\boldsymbol{x}$ and $\mathfrak{u}$, $|\mathfrak{u}| < \infty$,

$$(\boldsymbol{x}; \mathfrak{u}) = [y_1, y_2, \dots] \quad \text{with} \quad y_j = \begin{cases} x_j & \text{if } j \in \mathfrak{u}, \\ a & \text{otherwise.} \end{cases}$$

$(\boldsymbol{x}; \mathfrak{u})$ has $|\mathfrak{u}|$ active variables

For any finite $\mathfrak{u}$ and any $\boldsymbol{x} \in \mathcal{D}$, $L_{(\boldsymbol{x};\mathfrak{u})}(f) = f(\boldsymbol{x}; \mathfrak{u})$ is continuous,

$$\|L_{(\boldsymbol{x};\mathfrak{u})}\|^2 = \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \gamma_{\mathfrak{v}} \cdot K_{\mathfrak{v}}(\boldsymbol{x}, \boldsymbol{x}) < \infty.$$

Integration Problem

Given probability density unction ρ, approximate

$$\mathcal{I}(f) = \int_{\mathcal{D}} f(\boldsymbol{x}) \cdot \rho_\infty(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$= \lim_{d\to\infty} \int_{D^d} f(x_1,\dots,x_d,\boldsymbol{a}) \prod_{j=1}^{d} \rho(x_j)\,\mathrm{d}(x_1,\dots,x_d).$$

$\mathcal{I}$ is continuous iff

$$\sum_{\mathfrak{u}} \gamma_{\mathfrak{u}} \cdot C_0^{|\mathfrak{u}|} < \infty \quad \text{with} \quad C_0 = \int_D \int_D K(x,y)\cdot\rho(x)\cdot\rho(y)\,\mathrm{d}x\,\mathrm{d}y,$$

and for product weights $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$, iff

$$\prod_{j=1}^{\infty} (1+\gamma_j\cdot C_0) < \infty, \quad \text{i.e.,} \quad \sum_{j=1}^{\infty} \gamma_j < \infty$$

Algorithms: $\mathcal{A}_n(f) = \sum_{i=1}^{n} f(\boldsymbol{x}_i; \mathfrak{u}_i) \cdot a_i$

Errors:

$e^{\mathrm{wor}}(\mathcal{A}_n; \mathcal{F}) := \sup_{\|f\|_{\mathcal{F}} \le 1} |\mathcal{I}(f) - \mathcal{A}_n(f)|$ the worst case setting

$e^{\mathrm{ran}}(\mathcal{A}_n; \mathcal{F}) := \sup_{\|f\|_{\mathcal{F}} \le 1} \sqrt{\mathbb{E}|\mathcal{I}(f) - \mathcal{A}_n(f)|^2}$ the randomized setting

Cost of sampling $f(\boldsymbol{x}; \mathfrak{u})$: $\$(|\mathfrak{u}|)$, $\$$ is a cost function:

$\$: [0, \infty) \to [1, \infty)$ is monotonic

For instance cost of computing $f(x_1, a, x_3, a, a \ldots)$ equals $\$(2)$

Cost of $\mathcal{A}_n$: $\mathrm{cost}(\mathcal{A}_n) := \sum_{i=1}^{n} \$(|\mathfrak{u}_i|)$ as opposed to just n

COMPLEXITY AND TRACTABILITY

ε-Complexity: the minimal cost among algorithms with errors $\le \varepsilon$:

$$\mathrm{comp}^{\mathrm{sett}}(\varepsilon; \mathcal{F}) := \inf \left\{ \mathrm{cost}(\mathcal{A}) \ : \ e^{\mathrm{sett}}(\mathcal{A}, \mathcal{F}) \le \varepsilon \right\}$$

$\mathrm{sett} \in \{\mathrm{wor}, \mathrm{ran}\}$ in the worst case and randomized settings

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Polynomial Tractability: if there are C and p such that

$$\mathrm{comp}^{\mathrm{sett}}(\varepsilon,\mathcal{F}) \le C \cdot \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1)$$

The smallest such p is the exponent of tractability: p^{wor} and p^{ran}

Weak Tractability: if $\lim_{\varepsilon\to 0} \varepsilon \cdot \ln(\mathrm{comp}^{\mathrm{sett}}(\varepsilon;\mathcal{F})) = 0$

i.e., complexity is **NOT** exponential in $1/\varepsilon$

Example of Results: Classical Integration Problem

Based on Plaskota and W. [10]

Consider *Wiener kernel*

$$K(x,y) = \min(x,y) \quad \text{with} \quad D = [0,1] \quad \text{and} \quad \rho \equiv 1.$$

Then $a = 0$ and

$$\langle f, g \rangle_H = \int_0^1 f'(x) \cdot g'(x) \, \mathrm{d}x$$

Consider also product weights $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$ with

$$\gamma_j = \begin{cases} j^{-\beta} & \text{for } \beta > 1; \quad \text{polynomial case} \\ r^{-j} & \text{for } r \in (0,1); \quad \text{exponential case} \end{cases}$$

POLYNOMIAL WEIGHTS:

Polynomial Tractability with tractability exponents

$$p^{\rm wor} \le \max\left(1\,,\,\frac{2}{\beta-1}\right) \quad \text{and} \quad p^{\rm ran} \le \max\left(\frac{2}{3}\,,\,\frac{2}{\beta-1}\right)$$

even for cost function \$ as large as

$$\$(d) = \mathcal{O}\left(e^{k\cdot d}\right) \quad \text{for any } k \ge 0.$$

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OPTIMALITY:

Due to lower bound in Kuo,Sloan,W.,and Woźniakowski 09

$$p^{\mathrm{wor}} = \max\left(1\,,\,\frac{2}{\beta-1}\right) \quad \text{for any} \quad \Omega(1+d) \le \$(d) \le \mathcal{O}\left(e^{k\cdot d}\right)$$

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Weak Tractability even for $\$(d) = \mathcal{O}\left(e^{e^{k\cdot d}}\right)$

EXPONENTIAL WEIGHTS:

Polynomial Tractability with tractability exponent

$$p^{\mathrm{wor}} = 1 \quad \text{and} \quad p^{\mathrm{ran}} \leq \frac{2}{3}$$

even for cost function \$ as large as

$$\$(d) = \mathcal{O}\left(e^{k \cdot d^r}\right) \quad \text{for any } k \geq 0 \text{ and } r < 2$$

Weak Tractability even for $\$(d) = \mathcal{O}\left(e^{e^{k \cdot d^2}}\right)$

More General Results: From H to $\mathcal{F}$

Suppose that for the **univariate case**

$$\mathrm{comp}^{\mathrm{sett}}(\varepsilon; H) \le c \cdot \varepsilon^{-q} \cdot \$(1)$$

(For the Wiener case: $q = q^{\mathrm{wor}} = 1$ and $q = q^{\mathrm{ran}} = 3/2$)

For the **$d = \infty$ case:** (recall that $f = \sum_{\mathfrak{u}} f_{\mathfrak{u}}$)

$$\mathcal{A}_\varepsilon(f) = \sum_{\mathfrak{u} \in \mathfrak{U}(\varepsilon)} A_{\mathfrak{u}, n(\mathfrak{u},\varepsilon)}(f_{\mathfrak{u}}) \quad \text{with} \quad A_{\mathfrak{u}, n(\mathfrak{u},\varepsilon)}(f_{\mathfrak{u}}) = \sum_{k=1}^{n(\varepsilon,\mathfrak{u})} a_k \cdot f_{\mathfrak{u}}(\boldsymbol{x}_{k,\mathfrak{u}}; \mathfrak{u})$$

for specific $\mathfrak{U}(\varepsilon)$ and specific values of $n(\varepsilon, \mathfrak{u})$

Algorithms $A_{\mathfrak{u},n}$ obtained by Smolyak's construction for spaces $H_{\mathfrak{u}}$;
Smolyak [64] and W. and Woźniakowski [95]

Error:

$$e^{\text{sett}}(\mathcal{A}_\varepsilon; \mathcal{F}) \le \varepsilon$$

Cost:

The Largest Number of Active Variables:

$$d(\varepsilon) := \max_{\mathfrak{u} \in \mathfrak{U}(\varepsilon)} |\mathfrak{u}|$$

Since cost of computing $f_{\mathfrak{u}}(\boldsymbol{x}; \mathfrak{u})$ is $\le 2^{|\mathfrak{u}|} \cdot \$(|\mathfrak{u}|)$,
Kuo, Sloan, W., and Woźniakowski [09a]

we have

$$\text{cost}(\mathcal{A}_\varepsilon) \le 2^{d(\varepsilon)} \cdot \$(d(\varepsilon)) \cdot \sum_{\mathfrak{u} \in \mathfrak{U}(\varepsilon)} n(\mathfrak{u}, \varepsilon)$$

THM. 1 For polynomial weights $\gamma_j \le c \cdot j^{-\beta}$ with $\beta > 1$

$$d(\varepsilon) = o\left(\ln(1/\varepsilon)\right)$$

and

$$\mathrm{cost}(\mathcal{A}_\varepsilon) \le c_\delta \cdot \varepsilon^{-\max(q\,,\,2/(\beta-1)+\delta)} \cdot 2^{d(\varepsilon)} \cdot \$(d(\varepsilon))$$

Corollary 1 Polynomial tractability with tractability exponent

$$p^{\mathrm{sett}} \le \max\left(q\,,\,\frac{2}{\beta-1}\right) \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k\cdot d}\right)$$

Weak tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k\cdot d}}\right)$$

for any $k \ge 0$

THM. 2 For exponential weights $\gamma_j \le c \cdot r^j$ with $r \in (0,1)$

$$d(\varepsilon) = \mathcal{O}\left(\sqrt{\ln(1/\varepsilon)}\right)$$

and

$$\mathrm{cost}(\mathcal{A}_\varepsilon) \le C \cdot \varepsilon^{-q-\mathcal{O}(\ln(\ln(1/\varepsilon)))/\sqrt{\ln(1/\varepsilon)}} \cdot 2^{d(\varepsilon)} \cdot \$(d(\varepsilon))$$

Corollary 2 Polynomial tractability with tractability exponent

$$p^{\mathrm{sett}} \le q \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d^r}\right) \quad \text{for} \quad r < 2$$

Weak tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k \cdot d^2}}\right)$$

for any $k \ge 0$

From THM. 1, THM. 2 and Lower bound from Kuo, Sloan, W., and Woźniakowski

THM. 3 If $\lambda_j = \Theta\left(j^{-\beta}\right)$ $(\beta > 1)$ and if the exponent q is sharp then

$$p^{\mathrm{wor}} = \max\left(q, \frac{2}{\beta - 1}\right) \quad \text{as long as} \quad \Omega(d+1) \le \$(d) \le \Theta\left(e^{k\cdot d}\right).$$

If $\lambda_j = \Theta\left(r^j\right)$ and if the exponent q is sharp then

$$p^{\mathrm{wor}} = q \quad \text{as long as} \quad \Omega(d+1) \le \$(d) \le \Theta\left(e^{k\cdot d}\right).$$

RECALL: Exponent q is sharp iff $\mathrm{comp}^{\mathrm{sett}}(\varepsilon; H) = \Theta(\varepsilon^{-q})$

FUNCTION APPROXIMATION PROBLEM

Based on W. and Woźniakowski [10a] and [10b]

For $\mathcal{F}$ as before, approximate

$f = \sum_{\mathfrak{u}} f_{\mathfrak{u}}$ with error measured in L_2-type norm:

$$\|f\|_{\mathcal{G}}^2 = \sum_{\mathfrak{u}} \int_{D^{|\mathfrak{u}|}} |f_{\mathfrak{u}}(\boldsymbol{x})|^2 \cdot \rho_{\mathfrak{u}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

Problem is **well-defined** iff

$$\sup_{\mathfrak{u}} \gamma_{\mathfrak{u}}^{1/2} \cdot C_2^{|\mathfrak{u}|} < \infty, \quad \text{where} \quad C_2^2 = \sup_{\|f\|_H \le 1} \int_D |f(x)|^2 \cdot \rho(x) \, \mathrm{d}x$$

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Algorithms may now use two types of information:

standard: consisting of $f(\boldsymbol{x};\mathfrak{u})$ (as before), or

unrestricted linear: consisting of values $L(f)$ of linear functionals L

Selected results for UNRESTRICTED LINEAR INFORMATION

Based on W. and Woźniakowski [10a]

We do **NOT** need to assume that $K(a,a)=0$

Since L is continuous,

$$L(f) = \langle f, h \rangle_{\mathcal{F}}$$

Algorithms:

$$\mathcal{A}_n(f) = \sum_{k=1}^{n} \langle f, h_k \rangle_{\mathcal{F}} \cdot a_k,$$

where now $a_k \in \mathcal{F}$ are functions.

Errors: (worst case only) $e(\mathcal{A}_n;\mathrm{APP},\mathcal{F}) := \sup_{\|f\|_{\mathcal{F}}\le 1} \|f-\mathcal{A}_n(f)\|_{\mathcal{G}}$

Cost:

For $h=\sum_{\mathfrak{u}} h_{\mathfrak{u}}$, define **active variables of h** by

$$\mathrm{var}(h) = \bigcup_{\mathfrak{u}:h_{\mathfrak{u}}\neq 0} \mathfrak{u}$$

E.g., $\mathrm{var}(h_{\{1\}}+h_{\{1,2\}}+h_{\{1,2,3\}}) = \{1,2,3\}$

Then cost of computing $\langle f,h\rangle_{\mathcal{F}}$ equals $\$(|\mathrm{var}(h)|)$ and

$$\mathrm{cost}(\mathcal{A}_n) = \sum_{k=1}^{n} \$(|\mathrm{var}(h_k)|)$$

Complexity and Tractabilities defined as before.

From H to $\mathcal{F}$

Define $W : H \to H,$ $W(f)(x) = \int_D f(t) \cdot K(t,x) \cdot \rho(t)\, \mathrm{d}t$

ASSUMPTION (necessary even for $d = 1$): W is compact.

Let $(\lambda_j, \eta_j)_{j=1}^{\infty}$ be the eigenpairs of W,

$$W(\eta_j) = \lambda_j \cdot \eta_j, \quad \lambda_j \geq \lambda_{j+1}, \quad \text{and} \quad \langle \eta_j, \eta_k \rangle_H = \delta_{j,k}$$

For $\mathfrak{u} = \{u_1, \dots, u_{|\mathfrak{u}|}\}$ and $\boldsymbol{j} = [j_1, \dots, j_{|\mathfrak{u}|}]$ let

$$\lambda_{\boldsymbol{j},\mathfrak{u}} := \prod_{k=1}^{|\mathfrak{u}|} \lambda_{j_k} \quad \text{and} \quad \eta_{\boldsymbol{j},\mathfrak{u}}(\boldsymbol{x}) := \prod_{k=1}^{|\mathfrak{u}|} \eta_{j_k}(x_{\mathfrak{u}_k})$$

Given $\varepsilon > 0$, define $M(\varepsilon) := \left\{(\boldsymbol{j},\mathfrak{u}) \ : \ \gamma_{\mathfrak{u}} \cdot \lambda_{\boldsymbol{j},\mathfrak{u}} > \varepsilon^2\right\}$

THM. 4 Let

$$\mathcal{A}_\varepsilon^{\rm opt}(f) := \sum_{(\boldsymbol{j},\mathfrak{u})\in M(\varepsilon)} \langle f, \eta_{\boldsymbol{j},\mathfrak{u}}\rangle_{\mathcal{F}} \cdot \eta_{\boldsymbol{j},\mathfrak{u}}$$

We have

$$e^{\rm wor}(\mathcal{A}_\varepsilon^{\rm opt}; {\rm APP}, \mathcal{F}) \le \varepsilon \quad \text{and} \quad {\rm cost}(\mathcal{A}_\varepsilon^{\rm opt}) = \sum_{(\boldsymbol{j},\mathfrak{u})\in M(\varepsilon)} \$(|\mathfrak{u}|)$$

Moreover, $\mathcal{A}_\varepsilon^{\rm opt}$ **is optimal**, i.e.,

$${\rm comp}^{\rm wor}(\varepsilon; {\rm APP}, \mathcal{F}) = {\rm cost}(\mathcal{A}_\varepsilon^{\rm opt}) \le {\rm cost}(\mathcal{A})$$

for any $\mathcal{A}$ with error $\le \varepsilon$

$\mathcal{A}_\varepsilon^{\rm opt}$ is optimal for ALL cost functions \$

This enables to get

necessary and sufficient conditions

for polynomial and weak tractabilities

for different functions \$ and **general weights** $\gamma_{\mathfrak{u}}$

Illustration for product weights $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$

$$\gamma_j \le \begin{cases} j^{-\beta} \text{ for } \beta > 1; & \text{polynomial case} \\ r^{-j} \text{ for } r \in (0,1); & \text{exponential case} \end{cases}$$

THM. 5 For polynomial weights $\gamma_j \le c_1 \cdot j^{-\beta}$ with $\beta > 1$ and $\lambda_n \le c_2 \cdot n^{-\alpha}$ with $\alpha > 0$

$$d(\varepsilon) = o\left(\ln(1/\varepsilon)\right)$$

Hence:

Polynomial Tractability with tractability exponent

$$p \le 2/\min(\alpha, \beta) \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d^c}\right)$$

for $c \in (0, 1)$ and $k \ge 0$. This is sharp if the bounds on γ_j and λ_n are sharp.

Weak Tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k \cdot d}}\right)$$

THM. 6 For exponential weights $\gamma_j \le c \cdot r^j$ with $r \in (0,1)$ and $\lambda_n \le c_2 \cdot n^{-\alpha}$

$$d(\varepsilon) = 2 \cdot \sqrt{\frac{\ln(1/\varepsilon)}{\ln(1/r)}} \cdot (1 + o(1)) = \mathcal{O}\left(\sqrt{\ln(1/\varepsilon)}\right)$$

Hence:

Polynomial Tractability with tractability exponent

$$p \le 2/\alpha \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d^c}\right) \quad \text{for} \quad c < 2 \text{ and } k \ge 0$$

This is sharp if the bound on λ_n is sharp.

Weak Tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k \cdot d^c}}\right) \quad \text{for} \quad c < 2$$

Selected results for STANDARD INFORMATION

Based on W. and Woźniakowski [10b]

Algorithms: $\mathcal{A}_n(f) = \sum_{k=1}^{n} f(\boldsymbol{x}_k; \mathfrak{u}_k) \cdot a_k$

From H to $\mathcal{F}$

Suppose that for the **scalar case:**

$$\mathrm{comp}^{\mathrm{wor}}(\varepsilon; \mathrm{APP}, H) \leq c \cdot \varepsilon^{-q} \cdot \$(1)$$

Using Smolyak's construction we get algorithms $\mathcal{A}_\varepsilon^{\mathrm{std}}$ such that

$$e^{\mathrm{wor}}(\mathcal{A}_\varepsilon^{\mathrm{std}}; \mathrm{APP}, \mathcal{F}) \leq \varepsilon$$

and . . . . . .

THM. 7 For polynomial weights $\gamma_j \le c \cdot j^{-\beta}$ with $\beta > 1$

$$d(\varepsilon) = o(\ln(1/\varepsilon))$$

and

$$\mathrm{cost}(\mathcal{A}_\varepsilon^{\mathrm{std}}) = \mathcal{O}\left(\$(d(\varepsilon)) \cdot \varepsilon^{-\max(q, 2/\beta)}\right)$$

Hence: Polynomial Tractability with tractability exponent

$$p^{\mathrm{std}} \le \max(q, 2/\beta) \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d}\right)$$

and Weak Tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k \cdot d}}\right)$$

for $k \ge 0$.

THM. 8 For product weights $\gamma_j \le c \cdot r^j$ with $r \in (0,1)$

$$d(\varepsilon) = 2 \cdot \sqrt{\frac{\ln(1/\varepsilon)}{\ln(1/r)}} \cdot (1 + o(1)) = \mathcal{O}\left(\sqrt{\ln(1/\varepsilon)}\right)$$

and

$$\mathrm{cost}(\mathcal{A}_\varepsilon^{\mathrm{std}}) = \mathcal{O}\left(\$(d(\varepsilon)) \cdot \varepsilon^{-q-\delta}\right)$$

Hence: Polynomial Tractability with tractability exponent

$$p^{\mathrm{std}} \le q \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d^c}\right) \quad \text{for} \quad c < 2$$

$$p^{\mathrm{std}} \le q + \frac{4 \cdot k}{\ln(1/r)} \quad \text{even for} \quad \$(d) = \mathcal{O}\left(e^{k \cdot d^c}\right) \quad \text{for} \quad c = 2$$

and Weak Tractability even for

$$\$(d) = \mathcal{O}\left(e^{e^{k \cdot d^2}}\right) \quad \text{for} \quad k < \frac{\ln(1/r)}{4}$$

Final Comments

Complexity of many ∞-variate problems
is **not much higher** than
complexity of univariate problems over H

Smolyak's construction yields
efficient algorithms