Tractability of the Fredholm Problem of the Second Kind

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Fredholm problem

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$$q \in Q_{2d} \subseteq L_2(I^{2d})$$

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Here, d can be huge!!!

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- Dick, Kritzer, Kuo, Sloan (2007) polynomial tractability for weighted Korobov spaces

Algorithms and cost

- Algorithm:

$$A_{n,d}(f,q) = \phi\big(L_1(f), \dots, L_k(f), L_{k+1}(q), \dots, L_n(q)\big)$$

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- Cost:

$$\mathrm{cost}(A_{n,d}) = n + \text{arithmetic operations}$$

Complexity

- Worst case error:

$$e(A_{n,d}) = \sup_{\substack{f \in F_d \\ q \in Q_{2d}}} \|u_{f,q} - A_{n,d}(f,q)\|_{L_2(I^d)}$$

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- Weak tractability:

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n^{\mathrm{FP}\text{-}inf/com}(\varepsilon, d)}{\varepsilon^{-1}+d} = 0$$

Main Result: Information Complexity

- **Theorem:** Under some natural assumptions on $F = \{F_d\}_{d\in\mathbb{N}}$ and $Q = \{Q_{2d}\}_{d\in\mathbb{N}}$,

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- Approximation problem?

$$f \approx A_{n,d}(f) = \phi\big(L_1(f), \dots, L_n(f)\big)$$
$$e(A_{n,d}) = \sup_{f\in F_d} \|f - A_{n,d}(f)\|_{L_2(I^d)}$$
$$n^{\mathrm{APP}\text{-inf}}(\varepsilon, d) = \min\{\, n : \exists\, A_{n,d} \text{ with } e(A_{n,d}) \le \varepsilon \,\}$$

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- Extension property: For $q \in Q_d$, define

$$\begin{aligned} q_X(x,y) &= q(x) \\ q_Y(x,y) &= q(y) \end{aligned} \qquad \forall\, x, y \in I^d$$

Then $q_X, q_Y \in Q_{2d}$, with with

$$\|q_X\|_{Q_{2d}} \le \|q\|_{Q_d} \qquad \text{and} \qquad \|q_y\|_{Q_{2d}} \le \|q\|_{Q_d}.$$

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- Upper bound:
  $$\tilde{f} \approx f, \tilde{q} \approx q \implies u_{f,q} \approx u_{\tilde{f},\tilde{q}}$$

Example: Sobolev spaces

- Space $H_{d,m,\gamma}$:

$$\|v\|^2_{H_{d,m,\gamma}} = \sum_{\mathfrak{u}\subseteq\{1,2,\dots,d\}} \gamma_{d,\mathfrak{u}}^{-1} \int_{I^d} \left(\frac{\partial^{m|\mathfrak{u}|}}{\partial^m x_{\mathfrak{u}}} v(x) \right)^2 dx$$

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- $m_F > 1$ or $m_Q > 1$ implies intractability
- For $m_F = m_Q = 1$:
  - weak tractability: for all weights
  - quasi-polynomial tractability: for all weights
  - polynomial tractability: for decaying product weights

$$\gamma_{d,\mathfrak{u},F/Q} = \prod_{j\in\mathfrak{u}} \gamma_{d,j,F/Q} \qquad \text{with} \qquad \limsup_{d\to\infty} \frac{1}{\ln\, d} \sum_{j=1}^{d} \gamma^{\tau}_{d,j,F/Q} < \infty$$

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  - n_Q inner products to approximate q
- The solution $u_{\tilde{f},\tilde{q}}$ can be computed by solving

$$(\mathbf{I} - \mathbf{K})\mathbf{u} = \mathbf{b},$$

where $\mathbf{K}$ is an $m \times m$ contractive matrix, with

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- Can compute approximation of $u_{f,q}$ to within ε, with cost at most

$$\mathscr{O}\left(n^{\mathrm{FP\text{-}inf}}(\varepsilon, d) \ln \varepsilon^{-1}\right)$$

Conclusion

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- Total complexity? For tensor products, essentially the same as the information complexity.