Special Session

HAPPY BIRTHDAY STEFAN !!!!

Lower bounds for the complexity of linear functionals in the randomized setting

or how Aicke Hinrichs saved our work from 2002...

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The result of Aicke Hinrichs

Multivariate Integration :

$$I_d(f) \;=\; \int_{D_d} f(x)\,\rho_d(x)\,\mathrm{d}x \quad \textbf{for all} \quad f \in H(K_d)$$

with $D_d \subset \mathbb{R}^d$, ρ_d – **prob. density function** $\int_{D_d}\int_{D_d} K_d(x,y)\rho_d(x)\rho_d(y)\,\mathrm{d}x\,\mathrm{d}y < \infty$

Importance Sampling: ω_d – **prob. density function**

$$A_{n,d,\omega_d}(f) \;=\; \frac{1}{n}\sum_{j=1}^{n} f(x_j)\,\frac{\rho_d(x_j)}{\omega_d(x_j)}$$

$$e^{\mathrm{ran}}(A_{n,d,\omega_d}) \;=\; \sup_{\|f\|_{H(K_d)}\le 1} \left[\mathbb{E}_{\omega_d}\left(I_d(f) - A_{n,d,\omega_d}(f)\right)^2\right]^{1/2}$$

The result of Aicke Hinrichs

Theorem

- **If** $K_d(x,y) \ge 0 \quad \forall\, x,y \in D_d$ **then** $\exists\, \omega_d$

$$e^{\mathrm{ran}}(A_{n,d,\omega_d}) \le \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{\sqrt{n}}\; e^{\mathrm{ran}}(0)$$

- **For all** $\varepsilon \in (0,1)$ **and** $d \in \mathbb{N}$**, if**

$$n = \left\lceil \frac{\pi}{2} \left(\frac{1}{\varepsilon}\right)^2 \right\rceil$$

then

$$e^{\mathrm{ran}}(A_{n,d,\omega_d}) \le \varepsilon\, e^{\mathrm{ran}}(0)$$

Strong polynomial tractability with exponent ≤ 2 independently of the smoothness of functions from $H(K_d)$

Lower Bounds

Assumptions:

- $H(K_d)$ **tensor product,** $\rho_d(x) = \rho_1(x_1)\rho_1(x_2)\cdots\rho_1(x_d)$

$$K_d(x,y) = \prod_{j=1}^{d} K_1(x_j,y_j) \quad \forall\, x,y \in D_d = D_1^d$$

- K_1 **decomposable,** **Novak+W [2001],** $\exists\, a^* \in \mathbb{R}$

$$K_1(x,y) = 0 \quad x \le a^* \le y, \quad x,y \in D_1$$

- $D_{(0)} = \{x \in D_1 \mid \ x \le a^*\} \quad D_{(1)} = \{x \in D_1 \mid \ x \ge a^*\}$

$$I_1\Big|_{H(D_{(0)})} \neq 0 \quad \textbf{and} \quad I_1\Big|_{H(D_{(1)})} \neq 0$$

Main result

Let

$$n^{\mathrm{ran}}(\varepsilon, I_d) = \min\{\, n \mid \ \exists A_n \ \ e^{\mathrm{ran}}(A_n) \le \varepsilon\, e^{\mathrm{ran}}(0) \ \}$$

Then

$$n^{\mathrm{ran}}(\varepsilon, I_d) \ \ge \ \left\lceil \frac{1}{8} \left(\frac{1}{\varepsilon}\right)^2 \right\rceil \quad \forall\, \varepsilon \in (0,1) \quad d \ge \frac{2 \ln \varepsilon^{-1} - \ln 2}{\ln \alpha^{-1}}$$

and if $\mathbf{K_1 \ge 0}$

$$n^{\mathrm{ran}}(\varepsilon, I_d) \ \le \ \left\lceil \frac{\pi}{2} \left(\frac{1}{\varepsilon}\right)^2 \right\rceil \quad \forall\, \varepsilon \in (0,1) \ \ \forall\, d$$

with $\alpha \in [1/2, 1)$ depending on $H(K_1)$ and ρ_1.

Strong polynomial tractability with exponent 2

PS. How Aicke saved our work...

Example: Sobolev Spaces

Space:

$$H(K_1) = \left\{ f:\mathbb{R}\to\mathbb{R}\,|\ f(0)=f'(0)=\cdots f^{(r-1)}(0)=0, \quad f^{(r)}\in L_2(\mathbb{R}) \right\}$$

Inner Product: $\langle f,g\rangle_{H(K_1)} = \int_{\mathbb{R}} f^{(r)}(x)\,g^{(r)}(x)\,\mathrm{d}x$

Reproducing kernel:

$$K_1(x,y) = 1_M(x,y)\int_0^\infty \frac{(|x|-u)_+^{r-1}(|y|-u)_+^{r-1}}{[(r-1)!]^2}\,\mathrm{d}u,$$

where $1_M(x,y)=1$ **for** $xy\ge 0$, $1_M(x,y)=0$ **otherwise**

$$K_1 \ge 0, \quad K_1(x,y)=0 \quad \textbf{for} \quad x\le a^*=0\le y$$

Example: Sobolev Spaces

Gaussian integration: $I_d(f) = \int_{\mathbb{R}^d} f(x) \prod_{j=1}^d \frac{\exp(-x_j^2/2)}{\sqrt{2\pi}} \, \mathrm{d}x$

Randomized Setting:

$$n^{\mathrm{ran}}(\varepsilon, I_d) \quad \geq \quad \left\lceil \frac{1}{8} \left(\frac{1}{\varepsilon} \right)^2 \right\rceil \qquad \forall\, \varepsilon \in (0,1) \quad d \geq \frac{2 \ln \varepsilon^{-1} - \ln 2}{\ln 2}$$

$$n^{\mathrm{ran}}(\varepsilon, I_d) \quad \leq \quad \left\lceil \frac{\pi}{2} \left(\frac{1}{\varepsilon} \right)^2 \right\rceil \qquad \forall\, \varepsilon \in (0,1) \quad \forall\, d$$

for all smoothness parameters $r \geq 1$

Worst Case Setting: Novak + W [2001]

$$n^{\mathrm{wor}}(\varepsilon, I_d) \geq (1 - \varepsilon^2) 2^d \qquad \forall\, \varepsilon \in (0,1) \quad \forall\, d$$

Curse of Dimensionality

Example: Sobolev spaces

Asymptotic: d **fixed and** $\varepsilon \to 0$

$$\textbf{Randomized Setting}:$$

$$n^{\rm ran}(\varepsilon, I_d) \;=\; \Theta\left(\left(\frac{1}{\varepsilon}\right)^{1/(r+1/2)}\right)$$ **modulo logarithms**

$$\textbf{Worst Case Setting}:$$

$$n^{\rm wor}(\varepsilon, I_d) \;=\; \Theta\left(\left(\frac{1}{\varepsilon}\right)^{1/r}\right)$$

Quite different from tractability behavior

Example: Centered Discrepancy

Space:

$$H(K_1) = \left\{ f : [0,1] \to \mathbb{R} \mid \ f(\tfrac{1}{2}) = 0, \ \ f' \in L_2([0,1]) \right\}$$

Inner Product: $\langle f, g \rangle_{H(K_1)} = \int_0^1 f'(x)\, g'(x)\, \mathrm{d}x$

Reproducing kernel:

$$K_1(x,y) = 1_M(x,y)\, \min\left(|x - \tfrac{1}{2}|, |y - \tfrac{1}{2}| \right)$$

where $1_M(x,y) = 1$ **for** $(x - \frac{1}{2})(y - \frac{1}{2}) \ge 0$**,** $1_M(x,y) = 0$ **otherwise**

$$K_1 \ge 0, \qquad K_1(x,y) = 0 \quad \textbf{for} \quad x \le a^* = \tfrac{1}{2} \le y$$

Example: Centered Discrepancy

Uniform integration: $I_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x$

Randomized Setting:

$$n^{\mathrm{ran}}(\varepsilon, I_d) \geq \left\lceil \frac{1}{8}\left(\frac{1}{\varepsilon}\right)^2 \right\rceil \quad \forall\, \varepsilon \in (0,1) \quad d \geq \frac{2\ln \varepsilon^{-1} - \ln 2}{\ln 2}$$

$$n^{\mathrm{ran}}(\varepsilon, I_d) \leq \left\lceil \frac{\pi}{2}\left(\frac{1}{\varepsilon}\right)^2 \right\rceil \quad \forall\, \varepsilon \in (0,1) \ \ \forall\, d$$

Worst Case Setting: Novak + W [2001]

$$n^{\mathrm{wor}}(\varepsilon, I_d) \geq (1-\varepsilon^2)2^d \quad \forall\, \varepsilon \in (0,1) \quad \forall\, d$$

Curse of Dimensionality

Example: Centered Discrepancy

Asymptotic: d **fixed and** $\varepsilon \to 0$

$$\textbf{Randomized Setting}:$$

$$n^{\rm ran}(\varepsilon, I_d) \quad = \quad \Theta\left(\left(\frac{1}{\varepsilon}\right)^{2/3}\right) \quad \textbf{modulo logarithms}$$

$$\textbf{Worst Case Setting}:$$

$$n^{\rm wor}(\varepsilon, I_d) \quad = \quad \Theta\left(\left(\frac{1}{\varepsilon}\right)^{1}\right)$$

Quite different from tractability behavior

Generalizations

- **For decomposable kernels, lower bounds hold for all tensor product linear functionals I_d. However, upper bounds of Aicke Hinrichs hold, so far, only for multivariate integration...**
- **For kernels with a decomposable part, different lower bounds hold for all tensor product linear functionals I_d. However, they are not so sharp...**

Conclusions

- To obtain tractability, weighted spaces are not necessary for linear functionals in the randomized setting unlike in the worst case setting
- For unweighted spaces, the exponent of strong polynomial tractability does not depend on the smoothness of functions
- Weighted spaces are needed if we want to decrease the exponent of the strong polynomial tractability