

# Multilevel path simulation for jump-diffusion SDEs

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# Outline

- 1 Multilevel MC approach
- 2 Extension to jump-diffusion SDEs with constant jump rate
- 3 General jump-diffusion SDEs

# Multilevel Approach for geometric Brownian SDEs

Given a scalar SDE driven by a Brownian diffusion

$$dS(t) = a(S, t) dt + b(S, t) dW(t),$$

to estimate  $E[P] := E[f(S(T))]$  where the path-dependent payoff  $P$  can be approximated by  $\hat{P}_l$  using  $2^l$  uniform timesteps, we use

$$E[\hat{P}_L] = E[\hat{P}_0] + \sum_{l=1}^L E[\hat{P}_l - \hat{P}_{l-1}].$$

$E[\hat{P}_l - \hat{P}_{l-1}]$  is estimated using  $N_l$  simulations with same  $W(t)$  for both  $\hat{P}_l$  and  $\hat{P}_{l-1}$ ,

$$\hat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} (\hat{P}_l^{(i)} - \hat{P}_{l-1}^{(i)})$$

Using independent samples for each level, the variance of the combined estimator is

$$\mathbb{V} \left[ \sum_{l=0}^L \hat{Y}_l \right] = \sum_{l=0}^L N_l^{-1} V_l, \quad V_l \equiv \begin{cases} V[\hat{P}_l - \hat{P}_{l-1}], & l > 0 \\ V[\hat{P}_0], & l = 0 \end{cases}$$

and the computational cost is proportional to  $\sum_{l=0}^L N_l h_l^{-1}$

Hence, the variance is minimised for a fixed computational cost by choosing  $N_l$  to be proportional to  $\sqrt{V_l h_l}$ .

# Optimal computational cost of Multilevel MC Approach

Using Milstein discretization for a European call option, first order strong convergence gives

$$V_l \propto h_l^2$$

and hence total cost to achieve  $\varepsilon$  RMS error is  $O(\varepsilon^{-2})$  instead of usual  $O(\varepsilon^{-3})$ .

# Jump-diffusion SDEs

To capture the characteristics of fat-tail return distribution and the volatility smile effect, Merton introduced jump-diffusion SDEs to model stock price:

$$dS(t) = a(S(t-), t)dt + b(S(t-), t)dW_t + c(S(t-), t)dJ_t,$$

where the jump item  $J_t$  is a compound Poisson process

$\sum_{i=1}^{N(t)} (Y_i - 1)$ , the jump magnitude  $Y_i$  satisfies some probability distribution, and  $N(t)$  is a Poisson process with intensity  $\lambda$ , independent of the Brownian motion.

## A Jump-adapted Milstein discretisation

To do path simulation of jump-diffusion SDEs, the jump times need to be added into the discretisation time grid. We use the so-called jump-adapted approximation proposed by Platen in 1982 [Pla82], which sets the time grid as a superposition of jump times and the fixed-time grid.

$$\hat{S}_{n+1}^- = \hat{S}_n + a h_n + b \Delta W_n + \frac{1}{2} \frac{\partial b}{\partial S} b (\Delta W_n^2 - h_n) \quad ;$$

$$\hat{S}_{n+1} = \begin{cases} \hat{S}_{n+1}^- Y_i, & \text{when } t_{n+1} \in \mathbb{J} ; \\ \hat{S}_{n+1}^-, & \text{otherwise.} \end{cases}$$

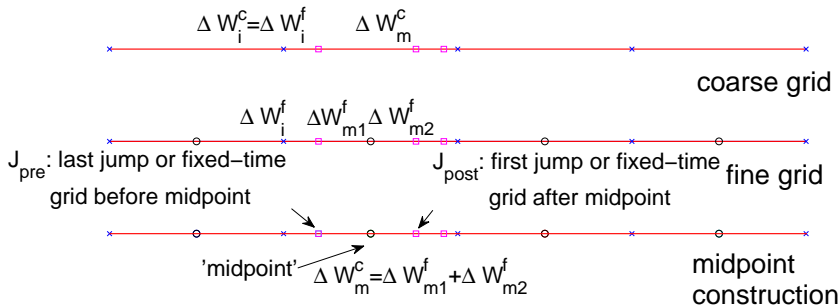
where  $\mathbb{J} = \{\tau_1, \tau_2, \dots, \tau_m\}$  is jump time set.

# Multilevel for jump diffusion when rate is constant

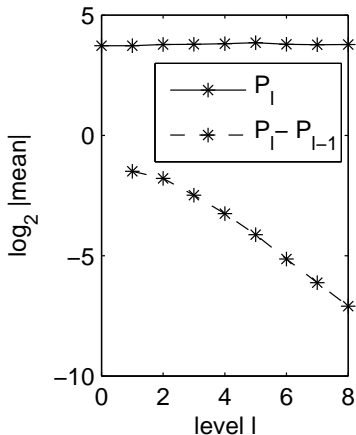
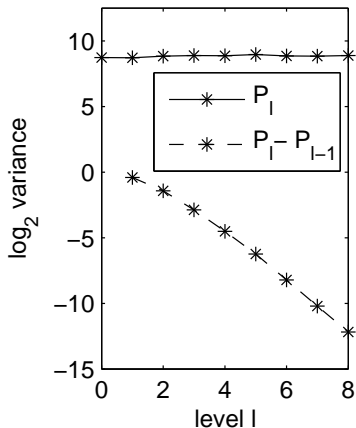
- use jump-adapted discretisation, adding jump times to standard uniform timestep discretisation times
- Milstein approximation of pure diffusion model between jumps, with conditional Brownian interpolation within each timestep for barrier and lookback options
- jump intervals are exponential random variables; the same values are used for coarse and fine paths



## "Midpoint" construction in multilevel approach



Numerical results for const rate jump diffusion European call,  
 $T=1$ ,  $S(0)=100$ ,  $r=0.05$ ,  $\sigma=0.2$ ,  
 $\log Y \sim N(-0.1, 0.2)$ ,  $\lambda = 2$ ;  $P = \exp(-rT) \max(S(T) - K, 0)$



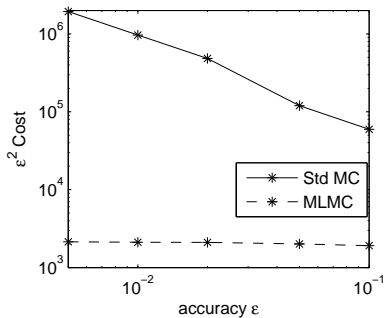
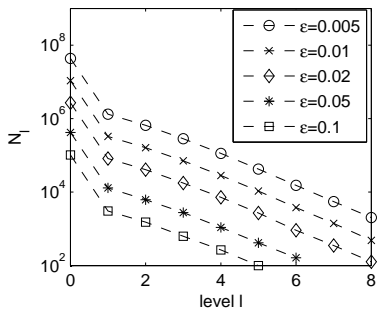


Figure: european vanilla option

## Numerical results for path-dependent options

option	numerics for jump-diffusion	analysis for plain diffusion
Lipschitz	$O(h^2)$	$O(h^2)$
Asian	$O(h^2)$	$O(h^2)$
lookback	$O(h^2)$	$o(h^{2-\delta})$
barrier	$O(h^{3/2})$	$o(h^{3/2-\delta})$
digital	$O(h^{3/2})$	$o(h^{3/2-\delta})$

Table:  $V_I$  convergence observed numerically for const rate jump-diffusion SDEs– matches results for standard diffusion (proved by Giles, Debrabant & Röbler), but numerical analysis is yet to be done.

# General jump-diffusion SDEs

There are several directions to generalize the Merton's model.

- Introduce dependency between parameters. As a particular example, we can let jump rate rely on the stock price, namely  $\lambda = \lambda(S(t-), t)$ , which is called state-dependent intensity.
- Replace the jump-diffusion SDE with SDE driven by Lévy processes which allows infinite frequency of jump within limited time (done by Dereich & Heidenreich in [DH10]).

The first case is relatively complex for Multilevel implementation because jump times differ in fine and coarse grids due to bias introduced to the approximated intensity by the discretisation scheme.

## Two approaches for state-dependent intensity

We can have two methods to cope with state-dependent intensity.

- Thinning method;  
In the literature, the case of bounded state-dependent intensity is discussed by Glasserman & Merener where they use a thinning approach to simulate the jump-diffusion SDEs exactly.
- Cumulative intensity method.  
For the unbounded case, we have the cumulative distribution function of jump times. Thus an inverse transform for the cumulative intensity can be adopted.

# Thinning method

The idea of thinning is use a constant rate Poisson process to generate candidate jump time, then accept them with a probability based on current jump rate.

By the terminology of marked point process, the jump part can be written as

$$\begin{aligned} \sum_{i=1}^{N(t)} f(Y_i) &= \int_{z \in E^*} f(S(t-), z) \mu(dz, dt) \\ &= \int_0^1 \int_{z \in E^*} f(S(t-), z) 1_A(u) \mu^*(dz, dt) du, \end{aligned}$$

$$\text{where } A = \left[0, \frac{\lambda(S(t-), t)}{\lambda_{\text{sup}}}\right], \quad 0 < t < T$$

in which  $\mu(\cdot, \cdot)$  is a random counting measure with intensity  $\lambda(S(t-), t)$ , and  $\mu^*(dz, dt)$  represents the random measure with intensity  $\lambda_{\text{sup}}$ .

# Thinning Algorithm

Hence we can have acceptance-rejection procedure:

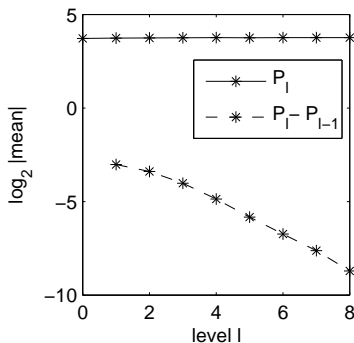
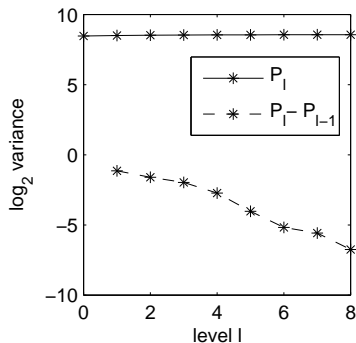
- 1 Generate the jump-adapted time grid using Poisson process with constant rate  $\lambda_{\text{sup}}$ ;
- 2 Simulate each interval of time grid using appropriate discretisation scheme, when the endpoint  $\tau$  is a jump time
- 3 Generate a uniform random number  $U \sim [0, 1]$ ,
  - 1 If  $p = \frac{\lambda(S(\tau-), \tau)}{\lambda_{\text{sup}}} > U$ , accept  $\tau$  add jump to the state value;
  - 2 Otherwise consider it as a normal grid point.



# Adopting multilevel

Thinning is easy to incorporate into Multilevel framework:  
Numerical result for European call

$$\lambda = \frac{\lambda_0}{1 + CS(t-)^2}, \quad \lambda_0 = 2, \quad C = 1e-4, \quad S_0 = 100$$



# Multilevel treatment

The variance convergence rate is approximately 1. What's wrong?

Jump candidates may not be accepted in both fine and coarse grids. Similar to the case of digital option, the proportion of those path where jumps differ is  $O(h)$ , giving an  $O(1)$  value for  $\hat{P}_I - \hat{P}_{I-1}$ , which leads to  $V_I = O(h)$ .

To improve convergence rate: we change the probability density under which the expectation is taken to force the fine and coarse path to jump simultaneously.

# Multilevel treatment

Write the estimator

$$\mathbb{E}_f[P_l] - \mathbb{E}_c[P_{l-1}] = \mathbb{E}_Q\left[P_l \prod_{\tau} R_f - P_{l-1} \prod_{\tau} R_c\right],$$

in the measure  $Q$ , the acceptance probability for a candidate jump is  $\frac{1}{2}$ . The Radon-Nikodym derivatives are

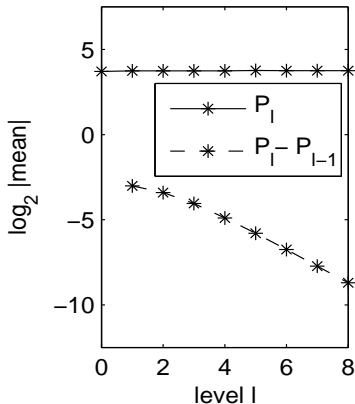
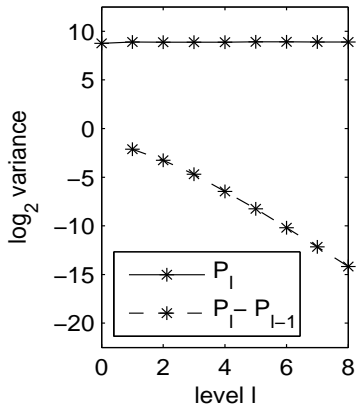
$$R_f = \begin{cases} 2p_f, & \text{if } U < \frac{1}{2}; \\ 2(1 - p_f), & \text{if } U \geq \frac{1}{2}, \end{cases} \quad R_c = \begin{cases} 2p_c, & \text{if } U < \frac{1}{2}; \\ 2(1 - p_c), & \text{if } U \geq \frac{1}{2}, \end{cases}$$

in which the probabilities are

$$p_f = \frac{\lambda^f(S(\tau-), \tau)}{\lambda_{\text{sup}}}, \quad p_c = \frac{\lambda^c(S(\tau-), \tau)}{\lambda_{\text{sup}}}.$$

## Numerical result, change of measure

$$\lambda = \frac{\lambda_0}{1 + CS(t-)^2}, \quad \lambda_0 = 2, \quad C = 1e-4 \quad S_0 = 100$$



## Numerical result, change of measure

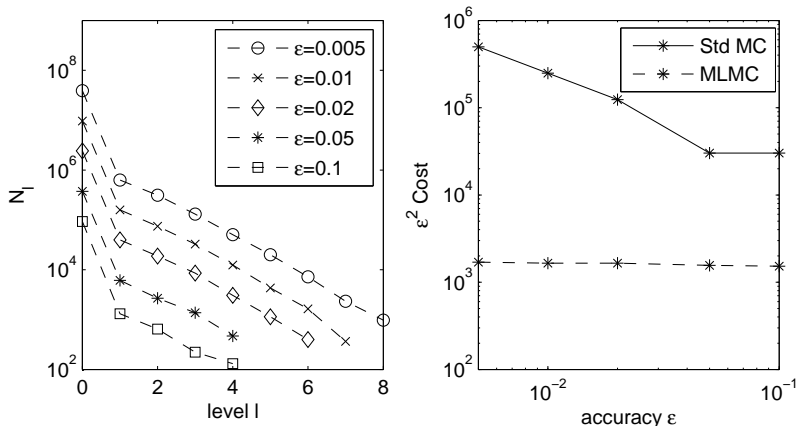


Figure: european vanilla option

# Conclusion

- Multilevel approach can be extended to scalar jump-diffusion SDEs by using jump-adapted schemes.
- The variance convergence order is maintained in the constant rate case.
- Extra amendment is needed in the state-dependent rate case by thinning and change of measure.
- Furthermore we have successfully used the cumulative intensity method to cope with the case of unbounded intensity.
- Plan to apply multilevel to variance gamma process.



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