On Quantization of Marginal Distributions of SDEs

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joint work with
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The Problem

Family of scalar autonomous SDEs

$$
\begin{aligned}
dX_t(x,a,b) &= a(X_t(x,a,b))\,dt + b(X_t(x,a,b))\,dW_t, \quad t \in [0,1] \\
X_0(x,a,b) &= x
\end{aligned}
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with $x \in \mathbf{I}$, $a \in \mathbf{A}$, $b \in \mathbf{B}$

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Approximate

$$S(x,a,b,f) = \mathbb{E}\, f(X_1(x,a,b)) = \int_{\mathbb{R}} f\, d\mu_{x,a,b}$$

for $x \in \mathbf{I}$, $a \in \mathbf{A}$, $b \in \mathbf{B}$, $f \in \mathbf{F}$

Deterministic algorithms based on initial value x and finitely many evaluations of a, b, f

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where $\widehat{\mu}_{x,a,b}$ is a discrete measure on $\mathbb{R}$

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Error and **cost** of $\widehat{S}$ at $(x,a,b,f) \in \mathbf{I} \times \mathbf{A} \times \mathbf{B} \times \mathbf{F}$

$$e(\widehat{S},(x,a,b,f)) = |S(x,a,b,f) - \widehat{S}(x,a,b,f)|$$

$$cost(\widehat{S},(x,a,b,f)) = \#\ \text{evaluations of}\ a,b,f + \#\ \text{arithmetical operations}$$

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$$e(\widehat{S}) = \sup_{x,a,b} \sup_{f \in \mathbf{F}} \Big| \int_{\mathbb{R}} f \, d\mu_{x,a,b} - \int_{\mathbb{R}} f \, d\widehat{\mu}_{x,a,b} \Big|$$

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Minimal errors

$$e_N = \inf\{e(\widehat{S}) : \ cost(\widehat{S}) \leq N\}$$

Result

Assumptions Let

$$\mathbf{I} = [-L, L]$$
$$\mathbf{A} = \mathbf{B} = \{h \in C^4(\mathbb{R}) : |h(0)|, |h^{(j)}| \leq K, j = 1, 2, 3, 4\}$$
$$\mathbf{F} = \mathbf{F}(M, \beta) = \{f \in C^4(\mathbb{R}) : |f^{(j)}(z)| \leq M \cdot (1 + |z|^\beta), j = 1, 2, 3, 4\}$$

with $L, K, M, \beta > 0$

Theorem For every $\delta > 0$

$$e_N \leq c \cdot N^{-\frac{1}{3+\delta}}$$

with $c = c(\delta, L, K, M, \beta)$

The Algorithm

Let $\delta < 1/2$. Fix $a \in \mathbf{A}, b \in \mathbf{B}, f \in \mathbf{F}$.

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leads to **Markov chain** $Y(x) = (Y_k(x))_{k \in \mathbb{N}_0}$ with

- initial value x
- state space D_x
- transition probabilities

$$p_{y,z} = \mathbb{P}(X^E(y) \in A_{y,z}), \quad y, z \in D_x,$$

where

$$X^E(y) = y + a(y) \cdot m^{-1} + b(y) \cdot m^{-1/2} \cdot V$$

with $V \sim \mathcal{N}(0,1)$

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Approximation

$$\widehat{S}(x, a, b, f) = \mathbb{E} f(Y_m(x)) = \sum_{z \in D_x} f(z) \cdot p_{x,z}^{(m)}$$

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$$\sup_{m\in\mathbb{N}} \max_{k\le m} \mathbb{E}|Y_k(y)|^j < \infty$$

- Small deviations of central moments. For $y \in D_x, j \in \mathbb{N}$

$$\rho_j(y) = |\mathbb{E}(X^E(y) - \mathbb{E}X^E(y))^j - \mathbb{E}(Y_1(y) - \mathbb{E}X^E(y))^j|$$

State space

$$D_x = \{x\} \cup D, \quad D = \{i \cdot m^{-1} : \; i = -m \cdot \lfloor m^{\delta} \rfloor, \ldots, m \cdot \lfloor m^{\delta} \rfloor\}$$

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Sets $A_{y,z}$. Take $A_{y,x} = \emptyset$ if $x \notin D$ and assume

$$\mathbb{R} = \bigcup_{z \in D} A_{y,z} \quad \text{with} \quad \begin{cases} A_{y,z} \subset (z - m^{-1}, z + m^{-1}), & z \neq \pm\lfloor m^\delta \rfloor \\ A_{y,z} \subset \left(z - m^{-1}, +\infty\right), & z = \lfloor m^\delta \rfloor \\ A_{y,z} \subset \left(-\infty, z + m^{-1}\right), & z = -\lfloor m^\delta \rfloor \end{cases}$$

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Then we get uniformly bounded moments and

$$\rho_j(y) \leq c \cdot (1 + |y|^c) \cdot m^{-(j+1)/2}$$

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Specific choice of the sets $A_{y,z}$ yields

$$\rho_j(y) \leq c \cdot (1 + |y|^c) \cdot m^{-\max((j+1)/2,\, 2)}$$

Error analysis (roughly)

Fix $x \in \mathbf{I}$. We have

$$\mathbb{E}f(X_1(x,a,b)) - \mathbb{E}f(Y_m(x)) = \sum_{k=1}^{m} \int_{\mathbb{R}} \Delta_k(y) d\mathbb{P}_{Y_{m-k}(x)}(y)$$

with

$$\Delta_k(y) = \mathbb{E}g_k(X_{1/m}(y,a,b)) - \mathbb{E}g_k(Y_1(y))$$

where

$$g_k = \mathbb{E}f(X_{(k-1)/m}(\cdot,a,b)) \in \mathbf{F}(c(M,K,\beta),\beta), \quad k = 1,\dots,m$$

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Taylor expansion of g_k at $z_y = \mathbb{E}X^E(y)$ and moment properties of $X_{1/m}(y,a,b) - z_y$ and $Y_1(y) - z_y$ imply

$$|\Delta_k(y)| \leq c \cdot (1 + |y|^\beta) \cdot \sum_{j=1}^{3} \kappa_j(y) + c \cdot (1 + |y|^c) \cdot m^{-2},$$

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$$\begin{aligned} \kappa_j(y) \leq & |\mathbb{E}(X_{1/m}(y,a,b) - z_y)^j - \mathbb{E}(X^E(y) - z_y)^j| \\ & + |\mathbb{E}(X^E(y) - z_y)^j - \mathbb{E}(Y_1(y) - z_y)^j| \end{aligned}$$

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By moment properties of $Y(x)$

$$\int_{\mathbb{R}} |\Delta_k(y)| \, d\mathbb{P}_{Y_{m-k}(x)}(y) \leq c \cdot m^{-2}$$

Hence

$$e(\widehat{S}, (x,a,b,f)) = |\mathbb{E}f(X_1(x,a,b)) - \mathbb{E}f(Y_m(x))| \leq c \cdot m^{-1}$$

Cost analysis

Recall

$$\widehat{S}(x, a, b, f) = \mathbb{E} f(Y_m(x)) = \sum_{z \in D_x} f(z) \cdot p_{x,z}^{(m)}$$

We have

	eval's of a, b	eval's of f	arithm. op.
$p_{y,z}$	$\leq c \cdot m^{1+\delta}$		$\leq c \cdot m^{2+2\delta}$
$p_{x,z}^{(m)}$			$\leq c \cdot m^{3+2\delta}$
quadrature formula		$\leq c \cdot m^{1+\delta}$	$\leq c \cdot m^{1+\delta}$

Hence,

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Related Questions and Results

- Quadrature on Wiener space

  Kusuoka (2001), Ninomiya (2003), Lyons, Victoir (2004), Kusuoka (2007), Ninomiya, Victoir (2008), ...

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  Kusuoka (2001), Ninomiya (2003), Lyons, Victoir (2004), Kusuoka (2007), Ninomiya, Victoir (2008), ...

- Nonlinear integration problems for SDE's

  Plaskota, Wasilkowski, Woźniakowski (2000), Kwas, Li (2003), Kwas (2005),

  Petras, Ritter (2006):
  fixed x, $b = 1$, f, varying a
  lower error bounds: for a of comparable smoothness

$$e_N \geq c \cdot N^{-2}$$

  with cost measured by # evaluations of a